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A
SHORT TREATISE
ON THE
PRINCIPLES
OF THE
DIFFERENTIAL
AND
INTEGRAL
CALCULUS.

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PREFACE.

SINCE the period at which the study of the Differential Calculus in its modern form has been cultivated in England, so numerous have been the treatises designed for the purposes of instruction in that calculus, that the appearance of a new work on the subject may be thought to need explanation.

The following short treatise, then, has been compiled with a special reference to the circumstances under which mathematical studies are at present prosecuted in Oxford. And it is hoped that the academical student will find it better adapted to his immediate wants and objects than the existing larger treatises, which, though most valuable sources of information on the higher parts of the calculus, and of illustration for all parts of it, are neither always sufficiently satisfactory in their exposition of the first principles, nor in their general design altogether well adapted to the purposes of elementary instruction.

In the present Compendium it has been the author's object to compress into the smallest compass, consistent with the full explanation of every process, an account of the *most essential principles* of the science. He has limited the extent of his selection to those parts which are at once of the most general and useful application, and of a nature suited to the earlier stages of a student's progress. In the Integral Calculus especially, he has carried the investigation only so far as to convey a general notion of its nature, and to supply what is requisite for its application to those subjects which usually form a part of an elementary course.

Throughout the whole, the author's attention has been exclusively directed to the elucidation of the fundamental principles of the science, rather than to the details of its applications: since it appears to him that it is in the former particular that most of the existing treatises are deficient, while in the latter they are redundant.

For a supply of examples, illustrations, and applications, the student is therefore presumed to have constantly at hand the valuable "Collection of Examples on the Differential Calculus," by Mr. Peacock; and for more general views of the science, and the discussion of its more abstruse departments, he is referred to the treatises of Lacroix, Boucharlat, Lardner, Jephson, and Hind, from which the fullest information may be derived. To these works particular references are given through all parts of the following Treatise.

For his materials the author makes no claim to originality, but acknowledges himself largely indebted to the treatises of Boucharlat and Lacroix. He is aware that the omission of examples may by many be regarded as a great defect. But should the present little work be favourably received, he hopes at a future period in some measure to remedy this defect by offering a collection of the applications of the calculus to the geometry of curves, with a view to the completion of his former introductory Treatise on the Elements of Curves.

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PART I.

THE PRINCIPLES

OF THE

DIFFERENTIAL CALCULUS.

THE PRINCIPLES

OF THE

DIFFERENTIAL CALCULUS.

Preliminary notions, and first principles of Differentiation.

THE first principles of the science which has been named the Fluxional or Differential Calculus are grounded upon a doctrine which, though it is often regarded as involving abstruse considerations, is yet in reality only an application of a simple algebraical truth, which we will proceed to demonstrate in the following manner:

From the nature of ratios, if we have

$$\frac{a}{b}=\frac{x}{y}=p,$$

we have also

$$\frac{a-x}{b-y}=p.$$

Now let $x=a$ and $y=b$; the first proportion is obviously preserved; and we also have in the second equation,

$$\frac{0}{0}=p,$$

in which value p is any finite quantity.

OBS. Thus if we obtain any expression which under particular conditions has *both its numerator and denominator reduced to* 0 *at the same time,* it is by no means to be imagined that the expression becomes nugatory or useless: it may have any finite value: and we shall afterwards find the means of assigning such values in particular cases, and shall have abundant proof of their utility.

This expression p, which is the value of the ratio when the numerator and denominator are both *absolutely* $=0$, is evidently the *limit* towards which the value of the ratio approaches, when its terms are *indefinitely diminished.*

Def. If any algebraic expression be given involving a quantity, supposed variable, in any of its powers, or combined by any operation with other quantities supposed constant, that expression has its changes of value solely dependent upon the value given to the variable; it is called a *function* of the variable, and is usually expressed by the following, or some similar notation;

$$y=f(x);$$

where f is not a coefficient, but the symbol of a function.

A function may evidently be of such a nature as either to vanish when the variable vanishes, or not.

Let the variable x receive an increment h; then the value which the function y assumes when x becomes $x+h$, bears a certain ratio to its former value dependent upon the particular form of the function.

Let these successive values be written,

$$y=f(x),$$
$$y_1=f(x+h).$$

Their difference, or

$$y_1-y=f(x+h)-f(x),$$

is the *increment of the function* corresponding to the *increment* h of the *variable* x.

Suppose we can obtain $f(x+h)$ in such a form that we have the *ratio* of the *increment* of the *function* to that of the *variable, in terms which do not vanish when* h *vanishes,*

$$\text{or } \frac{y_2 - y}{h} = m,$$

where m is any quantity which does not vanish with h.

When $h = 0$ let m become $= p$; but in this case $y_2 - y = 0$ also;

We have therefore the ratio of the increments reduced to the expression

$$\frac{0}{0} = p.$$

This is evidently the *limit* to which the ratio approaches when the increments are indefinitely diminished.

The quantity p, obtained for any particular function, is termed the *differential coefficient:* and is expressed by the notation

$$\frac{dy}{dx} = p.$$

Where d does not stand for a coefficient, but the symbol of an operation; implying that we have taken the *limit of the ratio of the increment of the function to that of the variable.*

The terms of this fraction are respectively called the *differentials* of y and of x.

If we had originally supposed x *decreasing*, or had taken $x - h$, we should have had the differential coefficient $= -p$.

To find the form of the quantity p, which will obviously be different according to the form of the given function, is the primary object of the Differential Calculus: the process is called differentiation, and is in most cases very easily effected.

Before proceeding to the consideration of those particular cases, we will first observe in general, that, In the case of any function *which can be differentiated,* if when $h=0$ we have

$$\frac{y_2-y}{h}=p,$$

When h is finite, we have already seen that the value of p becomes $=m$. Let this value $m=p+k$, k being either $=0$ or a function of h which vanishes with h. Thus *every function which can be differentiated must be capable of being expressed in the form*

$$y_2=y+ph+kh \quad . \; . \; . \; . \quad (1)$$

Also since in the differential coefficient, the differential of the *function* is that term with whose form we are concerned, the result is usually expressed *with reference to it,*

$$dy=pdx.$$

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

Let us take the very simple function $y=ax$.

Here $y_2=ax+ah$.

Whence $\frac{y_2-y}{h}=a$.

And when $h=0$, this becomes

$$\frac{dy}{dx}=a,$$

or, $dy=a\,.\,dx$

that is $d\,.\,ax=a\,.\,dx \quad . \; . \; . \; . \quad (2)$

Next let the function be

$$y=a+x.$$

Here $y_2=a+x+h$

and $\frac{y_2-y}{h}-1$

And when $h = 0$,

$$\frac{dy}{dx} = 1,$$

$$\text{or } dy = dx;$$

$$\text{that is, } d\,(a + x) = dx \quad \ldots\ldots \quad (3)$$

Thus we may observe, *a constant united by addition (or subtraction) disappears, but united by multiplication is retained, in the differentiation.*

To find the differential of a sum of several functions.

Let $v\ y\ z$, &c. be functions of x which can be differentiated separately, and let

$$\frac{dv}{dx} = p, \frac{dy}{dx} = q, \frac{dz}{dx} = r, \text{ \&c.}$$

these must have been derived from the forms

$$v_2 = v + ph + kh$$
$$y_2 = y + qh + lh$$
$$z_2 = z + rh + mh, \text{ \&c.}$$

Adding these together

$$v_2 + y_2 + z_2 = v + y + z + (p + q + r)\ h + (k + l + m)\ h.$$

Whence we have

$$\frac{v_2 + y_2 + z_2 - (v + y + z)}{h} = (p + q + r) + (k + l + m);$$

but since k, l, m, vanish with h, when $h = 0$, this becomes

$$= \frac{d\,(v + y + z)}{dx} = p + q + r$$

$$= \frac{dv}{dx} + \frac{dy}{dx} + \frac{dz}{dx},$$

or, suppressing the common denominator,

$$d\ (v + y + z) = dv + dy + dz \quad \ldots\ldots \quad (4)$$

Upon the same principle we may find the differential of a product of two or more functions.

Taking the expressions for $y_2\ z_2$ and multiplying them together, we have

$$\left.\begin{aligned} y_2\, z_2 = z\,[y + qh + lh] \\ + rh\,[y + \&c.\,] \\ + mh\,[y + \&c.] \end{aligned}\right\}$$

Whence, transposing, and dividing by h

$$\left.\begin{aligned} \frac{y_2 z_2 - yz}{h} = qz + ry + lz + my \\ + [rq + \&c.]h \end{aligned}\right\}$$

which, when $h = 0$, becomes

$$\frac{0}{0} = \frac{dyz}{dx} = qz + ry$$

or by substitution for q and r we have,

$$\frac{d\,.\,zy}{dx} = z\frac{dy}{dx} + y\frac{dz}{dx},$$

and suppressing the common factor dx,

$$d\,.\,zy = zdy + ydz.$$

Thus, to find the differential of the product of two variables, we must multiply each by the differential of the other, and add the products.

By means of this rule, we shall easily find the differential of a product of three variables.

Let it, for example, be yzu; make $yz = t$, whence we shall have $d\,.\,yzu = d\,.\,tu$.

But by what has preceded,

$$d\,.\,tu = tdu + udt,$$

and since $t = yz$, we have $dt = ydz + zdy$.

Substituting, therefore, these values of t and dt, the equation is changed into

$$d\,.\,yzu = yzdu + uydz + uzdy. \;.\;.\;.\;.\;. \quad (5)$$

We see then that the same rule still holds for a product of three variables; viz. *that we must write down the product* yzu, *replace successively each variable by its differential, and add the products.*

The same rule evidently holds good for any greater number of variables.

The differential of a fraction $\frac{z}{y}$ is $\frac{ydz - zdy}{y^2}$; for suppose $\frac{z}{y} = t$, we have $z = ty$, and $dz = tdy + ydt$ (form 5), from which we find $ydt = dz - tdy$: putting on the second side the value of t, there results,

$$ydt = dz - \frac{z}{y}dy,$$

reducing to the same denominator,

$$ydt = \frac{ydz - zdy}{y},$$

and lastly,

$$dt = \frac{ydz - zdy}{y^2}, \text{or } d \, . \frac{z}{y} = \frac{ydz - zdy}{y^2} \, . \, . \, . \, . \, . \, . \quad (6)$$

If in the equation $d \, . \, yzu = yzdu + yudz + uzdy$, we divide each term by yzu, we shall get

$$\frac{d.yzu}{yzu} = \frac{du}{u} + \frac{dz}{z} + \frac{dy}{y};$$

and generally, by dividing the differential of the product of any number of variables by the product itself, we shall find,

$$\frac{d.xyztu \, . \, . \, . \, . \, .}{xyztu \, . \, . \, . \, . \, .} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + \frac{dt}{t} + \frac{du}{u} + \text{\&c.}$$

If now x, y, z, t, u, &c., be each equal to x, and the number of them be m, we shall have on the second side of this equation m terms, each equal to $\frac{dx}{x}$; the second side will therefore be changed into $\frac{mdx}{x}$, and the equation will become

$$\frac{d.x^m}{x^m}=\frac{mdx}{x},$$

and multiplying by x^m, we shall have

$$d.x^m=mx^{m-1}dx, \quad . \; . \; . \; . \; . \quad (7)$$

We may hence deduce this rule: *when a variable is raised to a power* m, *to obtain its differential, we must* 1st. *make the index the coefficient;* 2d. *diminish the index of the variable by unity;* 3d. *multiply this product by* dx.

The same rule will hold, if the index be fractional or negative, as may be thus shewn:

To prove the first case, let $y=x^{\frac{p}{q}}$; raise both sides to the power q, when we shall have $y^q=x^p$, and therefore, by form (7), $qy^{q-1}dy=px^{p-1}dx$; whence we find,

$$dy=\frac{p}{q}.\frac{x^{p-1}}{y^{q-1}}dx;$$

and since x^{p-1}, y^{q-1}, may be put under the forms $\frac{x^p}{x},\frac{y^q}{q}$, substituting these values, we have

$$dy=\frac{p}{q}.\frac{x^p}{y^q}.\frac{y}{x}dx;$$

and since $x^p=y^q$, the preceding equation is reduced to

$$dy=\frac{p}{q}\frac{y}{x}dx;$$

Lastly, putting for y its value, we obtain

$$dy=\frac{p}{q}.\frac{x^{\frac{p}{q}}}{x}dx;$$

and bringing the denominator x into the numerator, we have

$$dy=\frac{p}{q}.x^{\frac{p}{q}-1}dx,$$

a result the same as would have been obtained for the differential of $y = x^{\frac{p}{q}}$ by applying the form 7.

To demonstrate the case in which the index is negative, let $y = x^{-p}$: this is the same with $y = \frac{1}{x^p}$, which being differentiated by the rule for fractions, we shall have

$$dy = \frac{x^p d.1 - 1.dx^p}{x^p \times x^p}.$$

Observing that unity being a constant, its differential is 0, this expression is reduced to $dy = -\frac{d.x^p}{x^{2p}}$; whence differentiating by form (7) we shall have

$$dy = -\frac{px^{p-1}dx}{x^{2p}},$$

and subtracting the index $2p$ from the index $p-1$, there results, lastly, $dy = -px^{-p-1}dx$, as we should have found by applying the form 7. We conclude, therefore, that this rule holds true, whatever be the index of x, that is to say, whether the index be integral, fractional, positive or negative.

DIFFERENTIATION OF EXPONENTIAL FUNCTIONS.

To differentiate the exponential function

$$y = a^x,$$

when x becomes $x + h$, we have

$$y_2 = a^{x+h},$$

and the difference or increment is

$$a^{x+h} - a^x = a^x (a^h - 1).$$

In order to express it in terms involving h, as before explained, (form 1,) we may assume $a = 1 + b$, whence

$$a^h = (1 + b)^h = 1 + hb + \frac{h(h-1)}{2} b^2$$
$$+ \frac{h.(h-1)(h-2)}{2.3} b^3 + \&c.$$

$$\text{or} \quad a^h - 1 = hb + \frac{h(h-1)}{2} b^2 + \&c.$$

$$\therefore a^x (a^h - 1) = a^x \left(hb + \frac{h(h-1)}{2} b^2 + \&c.\right)$$

or arranging by powers of h, this

$$= a^x \left\{ \begin{array}{l} h\left(\frac{b}{1} - \frac{b^2}{2} + \&c.\right) \\ + h^2\left(\frac{b^2}{2} - \&c.\right) \end{array} \right.$$
$$+ \&c.$$

$$\therefore \frac{a^{x+h} - a^x}{h} = a^x \left(\frac{b}{1} - \frac{b^2}{2} + \&c.\right) + \&c.$$

or when $h = 0$, (writing the constant part $= A$)

$$\frac{da^x}{dx} = Aa^x$$

$$\text{or} \quad dx = \frac{1}{A} \frac{dy}{y} \quad \ldots\ldots \quad (8)$$

DIFFERENTIATION OF TRIGONOMETRICAL FUNCTIONS.

From geometrical considerations it appears that the limit of the ratio of the sine to the arc is unity; or we have in the case of the limit, $\frac{\sin. h}{h} = 1$.

To find the differential of the sine whose arc is x, suppose that the arc receives an increment h; then we have, by trigonometry,

$$\sin.(x+h) = \sin. x . \cos. h + \cos. x . \sin. h.$$

Subtracting from this function its primitive, and dividing by the increment h of the variable, we shall have

$$\frac{\sin.(x+h) - \sin. x}{h} = \frac{\sin. x . \cos. h + \sin. h . \cos. x - \sin. x}{h};$$

and collecting together the terms multiplied by $\sin. x$ on the second side, we shall find

$$\frac{\sin.(x+h) - \sin. x}{h} = \frac{\sin. x(\cos. h - 1)}{h} + \frac{\sin. h . \cos. x}{h}.$$

When h becomes 0, $\cos. h - 1$ becomes also 0, and $\frac{\cos. h - 1}{h}$ is reduced to the form $\frac{0}{0}$; it is necessary, therefore, to put that term under some other form, and, for this purpose, the equation $\cos.^2 h + \sin.^2 h = 1$, gives us $\cos.^2 h - 1 = -\sin.^2 h$, or $(\cos. h - 1)(\cos. h + 1) = -\sin.^2 h$; from which we get

$$\cos. h - 1 = -\frac{\sin.^2 h}{\cos. h + 1};$$

and substituting this value in the former equation, it becomes

$$\frac{\sin.(x+h) - \sin. x}{h} = -\frac{\sin. x . \sin.^2 h}{h(\cos. h + 1)} + \cos. x . \frac{\sin. h}{h}$$

When $h = 0, \frac{\sin. h}{h} = 1, \frac{\sin.^2 h}{\cos. h + 1} = \frac{0}{2} = 0$;

and therefore the equation is reduced to

$$\frac{d . \sin. x}{dx} = \cos. x;$$

whence we deduce $d\,.\sin.\,x = \cos.\,x\,.\,dx$. . . . . . . (9)

In this demonstration, the radius has been supposed unity. If we wish to have the differential of a sine whose radius is a, we must make use of this form;

$$\sin.\,(x+h) = \frac{\sin.\,x\,.\cos.\,h + \sin.\,h\,.\cos.\,x}{a};$$

and therefore, in the preceding result, it will be necessary to introduce the constant a, which will give

$$d\,.\sin.\,x = \frac{dx\,.\cos.\,x}{a},$$

for the differential of the sine of an arc whose radius is a.

This expression for the differential of sin. x being obtained, those for the differentials of the other trigonometrical lines very easily follow.

To find the differential of cos. x, the equation, $\sin.^2\,x + \cos.^2\,x = 1$, or rather, $(\sin.\,x)^2 + (\cos.\,x)^2 = 1$, being differentiated, gives $2\sin.\,x.d\,.\sin.\,x + 2\cos.\,x.d\,.\cos.\,x = 0$; whence we find,

$$d\,.\cos.\,x = -\frac{\sin.\,x.d.\sin.\,x}{\cos.\,x};$$

and putting for $d\,.\sin.\,x$ its value, $dx\,.\cos.\,x$, by (9), and reducing, we have $d\,.\cos.\,x = -dx\,.\sin.\,x$. . . . . (10)

We obtain the differential of tangent x, by considering that $\tan.\,x = \dfrac{\sin.\,x}{\cos.\,x}$; which equation being differentiated by form 6, we find

$$d\,.\tan.\,x = \frac{\cos.\,x\,.\,d\,.\sin.\,x - \sin.\,x\,.\,d\,.\cos.\,x}{\cos.^2\,x};$$

and putting the values of $d\,.\sin.\,x$ and $d\,.\cos.\,x$, we shall have $d\,.\tan.\,x = \left(\dfrac{\cos.^2\,x + \sin.^2\,x}{\cos.^2\,x}\right)dx$, and therefore, since $\cos.^2 x + \sin.^2 x = 1$,

$$d\,.\tan.x = \frac{dx}{\cos.^2 x}. \ldots\ldots\ldots\ldots \quad (11)$$

We know, by trigonometry, that the radius is a mean proportional between the tangent and the cotangent, and between the cosine and the secant, which gives

$$\cot.x = \frac{1}{\tan.x}, \quad \sec.x = \frac{1}{\cos.x}.$$

By differentiating the first of these equations (form 6,) we find

$$d\,.\cot.x = -\frac{d\,.\tan.x}{\tan.^2 x} = -\frac{dx}{\cos.^2 x\,.\tan.^2 x} = -\frac{dx}{\sin.^2 x}; \ldots \quad (12)$$

for, from the equation $\frac{\sin.}{\cos.} = \tan.$, we deduce

$\cos.\tan. = \sin.$

The equation $\sec.x = \frac{1}{\cos.x}$, being differentiated, gives

$$d\,.\sec.x = -\frac{d\,.\cos.x}{\cos.^2 x} = \frac{\sin.x dx}{\cos.^2 x} = \frac{\sin.x}{\cos.x}.\frac{dx}{\cos.x}$$

$$= \tan.x\,.\sec.x dx. \ldots\ldots\ldots\ldots \quad (13)$$

We may determine in the same manner the differential of the cosecant; for $\operatorname{cosec}.x = \frac{1}{\sin.x}$; which, being differentiated, we have

$$d\,.\operatorname{cosec}.x = -\frac{\cos.x dx}{\sin.^2 x} = -\frac{\cos.x}{\sin.x}.\frac{dx}{\sin.x.}$$

$$= -\frac{1}{\tan.x}\,.\operatorname{cosec}.x dx = -\cot.x \operatorname{cosec}.x dx. \ldots \quad (14)$$

For the versed sine, by differentiating the equation versin. $x+\cos. x=1$, we find d. versin. $x=d\,(1-\cos. x)$, and by performing the differentiation,

$$d.\,\text{versin.}\,x=\sin.\,xdx. \,\ldots\ldots \quad (15)$$

ON SUCCESSIVE DIFFERENTIATION.

Let y be a function of x; if this be differentiated, we shall find a result of the form pdx, p being a quantity which may be a function of x: in this case we shall be able to differentiate p also, and so obtain a result represented by qdx; proceeding in the same manner with respect to q, the result will be of the form rdx, and so on: pdx, qdx, rdx, &c. are the successive differentials of y.

The equations

$$dy=pdx \text{ give, dividing by } dx,\ \frac{dy}{dx}=p,$$

$$dp=qdx \ \ldots\ldots\ldots\ldots \ \frac{dp}{dx}=q,$$

$$dq=rdx \ \ldots\ldots\ldots\ldots \ \frac{dq}{dx}=r;$$

&c. &c.

q being obtained by two successive differentiations, and by dividing each time by dx, we will represent the operation by $\frac{d^2y}{dx^2}$, and we shall have $\frac{d^2y}{dx^2}=q$; in like manner by differentiating anew, and dividing by dx, we have $\frac{d^3y}{dx^3}=r$; and so on.

dy is the first differential of y;
d^2y is the second differential;
d^3y is the third differential;

and so on.

It may happen that a certain number of differentiations shall produce a differential which is no longer a function of x; in which case the process cannot be carried any further.

ON

THE DEVELOPEMENT

OF

FUNCTIONS.

Having thus established the principles on which any functions can be differentiated, and explained the nature of successive differentiation, we proceed to investigate the form in which a function may be developed in a series.

If we have a function of x in this form,

$$f(x) = A + Bx + Cx^2 + Dx^3 + \&c.$$

When x becomes $x+h$, this becomes

$$f(x+h) = A + B\ (x+h) + C\ (x+h)^2 + \ \&c.$$

or multiplying and expanding, it

$$= \begin{cases} A + Bx + Bh \\ + Cx^2 + 2\ Cxh + Ch^2 \\ + Dx^3 + 3\ Dx^2h + 3\ Dxh^2 + Dh^3 \\ + \&c. \end{cases}$$

which, arranging and collecting the terms,

$$= \begin{cases} A + Bx + Cx^2 + Dx^3 + \&c. \\ + (B + 2\ Cx + 3\ Dx^2 +)\ h \\ + (C + 3\ Dx + \ \&c. \qquad)\ h^2 \\ + (D + \&c. \qquad\qquad\qquad)\ h^3 \\ \qquad \&c. \end{cases}$$

Or it may be written, for brevity,

$$= \alpha + \beta\ h + \gamma\ h^2 + \delta\ h^3 + \&c.$$

Here α is evidently the original value of the function, when $h=0$, or

$$\alpha = f(x) = y$$

If we differentiate that expression, and divide by $d\ x$, we have

$$\frac{dy}{dx} = B + 2\ Cx + 3\ Dx^3 + \&c.$$

Whence differentiating successively,

$$\frac{d^2y}{dx^2} = 2\,C + 2\,.\,3\,Dx + \&c.$$

$$\frac{d^3y}{dx^3} = 2\,.\,3\,D + \&c.$$

$$\&c. \qquad \&c.$$

But comparing these results with the coefficients in the developement of $f\,(x+h)$ we find

$$\frac{dy}{dx} = \beta$$

$$\frac{d^2y}{dx^2} = 2\gamma \qquad \therefore \gamma = \frac{d^2y}{dx^2}\,.\,\frac{1}{2}$$

$$\frac{d^3y}{dx^3} = 2\,.\,3\,.\,\delta \qquad \therefore \delta = \frac{d^3y}{dx^3}\,.\,\frac{1}{2\,.\,3}$$

$$\&c. \qquad \&c.$$

Whence we have the developement

$$f(x+h) = fx + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\,.\,\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\,.\,\frac{h^3}{1.2.3} + \&c.\,.\quad(16)$$

In this formula we have obtained an expression for the developement of $f\,(x+h)$ in a series of ascending powers of h, with coefficients involving the successive differential coefficients of $f\,(x)$ derived from the assumption that $f\,(x)$ could be expressed by a series of ascending powers of x.

But since the values of those differential coefficients must be the same in whatever way they are deduced, it follows that if we have $f\,(x)$ *in any form such, that we can obtain its successive differential coefficients,* these will give us the coefficients of a series of ascending powers of h, divided respectively by the numbers, 1, 1 . 2, 1 . 2 . 3, &c. which will represent the developement of the function when x becomes $(x+h)$.

This formula is called Taylor's Theorem, from the name of the inventor, Dr. Brook Taylor.

The successive differentiations above performed on the series expressing $f(x)$, when x becomes $=0$, will have the following values, which we will write, expressing the condition $x=0$ by enclosing the differential coefficients in brackets:

$$\left(\frac{dy}{dx}\right)=B$$

$$\left(\frac{d^2y}{dx^2}\right)=2\,C$$

$$\left(\frac{d^3y}{dx^3}\right)=2\,.\,3\,.\,D$$

Or we have the series for $f(x)$

$$f(x)=(f(x))+\left(\frac{dy}{dx}\right)x+\left(\frac{d^2y}{dx^2}\right)\frac{x^2}{1\,.\,2}+\&c.\;\ldots\ldots \quad (17)$$

This formula gives us the developement of $f(x)$ in ascending powers of x, and the successive differential coefficients in the form which they take when $x=0$.

This is commonly called Maclaurin's Theorem; but is undoubtedly the invention of Stirling.

This formula necessarily supposes that $f(x)$ can be expressed by a series of powers of x.

In regard to Taylor's Theorem, it may be necessary to observe that the developement of the function in this form supposes the successive differential coefficients to be finite quantities, or if at any term the corresponding differential coefficient should become infinite, the developement could not be carried further.

These remarks are made, because some writers, attempting to apply the formulæ in cases of this kind, have called them failures or exceptions. It is not necessary here to enter further upon the subject; but the reader may consult Boucharlat (Transl.) p. 162. or Lardner's Diff. Calc. p. 43, 49.

These theorems for the developement of functions are however only particular cases of formulæ yet more general; viz. those of La Grange and La Place. For an account of these, which would be unsuitable to so elementary a treatise as the present, see Lardner, p. 50. or Lacroix, (Transl.) note E.

◆

ON MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE.

We may, in the series of Taylor, give such a value to the increment *h*, that any one of the terms of the series shall be greater than the sum of all those that follow. For, the series being represented by

$$y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.,$$

if we wish that $\frac{dy}{dx}h$, for instance, should become greater than the sum of all the other terms following, we may write the part of the series commencing from that term, thus:

$$\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\frac{h}{1.2}+\frac{d^3y}{dx^3}\frac{h^2}{1.2.3}+\&c.\right)h\,;$$

and since, when we make $h=0$, the part

$$\frac{d^2y}{dx^2}\frac{h}{2}+\frac{d^3y}{dx^3}\frac{h^2}{2.3}+\&c.$$

vanishes, it will be easily seen that by taking *h* exceedingly small, that part may be made as small as we please, and therefore be made less than $\frac{dy}{dx}$, which is independent of h. Let z be what $\frac{d^2y}{dx^2}.\frac{h}{2}+$ &c. becomes in this case: then the series will be reduced to $\left(\frac{dy}{dx}+z\right)h$; and since we have $\frac{dy}{dx}>z$, or, multiplying by h, $\frac{dy}{dx}h>zh$, it follows that the term $\frac{dy}{dx}h$ is greater than the sum of all the succeeding terms. The same may be proved for every term in respect of those that follow.

This being premised, if we have an expression for any function, as $y=fx$, and if at any particular value of y, x be supposed to receive an increment h, and then an equal decrement, and we have the conditions fulfilled,

$$f(x+h)<fx, \qquad f(x-h)<fx.$$

Then this value of y is said to be a maximum value of the function.

If, on the contrary, we have the conditions,

$$f(x+h)>fx, \qquad f(x-h)>fx.$$

The value of y is then said to be a *minimum*.

Or, when $f(x+h)$ and $f(x-h)$ are at the same time *both less* than fx, there will be a *maximum*; and if these functions be at the same time *both greater* than fx, there will be a *minimum*: if, lastly, *one* of these functions be *greater* and the *other less*, than fx, there will be *neither* a maximum nor a minimum.

We must therefore investigate the cases in which these conditions can be fulfilled; and for this purpose we have, by Taylor's theorem,

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3}\text{&c.}$$

in which series, if we change h into $-h$, we find also

$$f(x-h)=y-\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

In order, therefore, that $y=fx$ may be a maximum or a minimum, these two developements must be both less or both greater than y; but this cannot be, unless $\frac{dy}{dx}$ be $=0$. For by giving to h an exceedingly small value, we may always render $\frac{dy}{dx}h$ greater than the sum of all the terms that follow; in which case the sign of $\frac{dy}{dx}h$ will be likewise the sign of $\frac{dy}{dx}h$, together with the following terms; so that, on this hypothesis, if $\frac{dy}{dx}h$ be positive in one of the above developements, that developement will be greater than y, and will be less than y, if $\frac{dy}{dx}h$ be negative. But the signs of $\frac{dy}{dx}h$ are different in these developements, and therefore, if $\frac{dy}{dx}h$ be positive in one, it must be negative in the other: whence it follows, that one of the quantities, $f(x+h)$ and $f(x-h)$, will be greater, and the other less than fx.

If, therefore, $\frac{dy}{dx}$ be not $=0$, there cannot be either a maximum or a minimum; but if $\frac{dy}{dx}=0$, then the developements will be reduced to

$$\left.\begin{aligned} f(x+h)&=y+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\&c.,\\ f(x-h)&=y+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.;\end{aligned}\right\}\quad\ldots(18)$$

in which case, the sign of the terms that follow y will depend on $\frac{d^2y}{dx^2}$, if only h be taken so small that that term may be greater than the sum of all those that follow; and since $\frac{d^2y}{dx^2}$ has the same sign in the two developements, it follows, that if $\frac{d^2y}{dx^2}$ be positive, the two functions of $x+h$ and $x-h$ will be both greater than fx; and in this case, therefore, fx will be a minimum. In the same manner it will be seen, that if $\frac{d^2y}{dx^2}$ be negative, fx will be a maximum.

The principle, however, is more general.

If, besides $\frac{dy}{dx}=0$, we have also $\frac{d^2y}{dx^2}=0$, in this case we cannot have a maximum or minimum, unless also $\frac{d^3y}{dx^3}=0$. Then, taking h exceedingly small, the sign of the quantities following y will depend on $\frac{d^4y}{dx^4}$, and we may prove, as before, that if $\frac{d^4y}{dx^4}$ be positive, fx is a minimum; and if $\frac{d^4y}{dx^4}$ be negative, fx is a maximum; and so on. Generally, *if the first coefficient, that does not vanish, be of an even order, there will be a minimum when it is positive, and a maximum when it is negative.*

By these conditions then we may in any given function find the value of x, which gives y a maximum or minimum.

DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES.

If we have a function of two independent variables, as

$$u=f(x,y),$$

its variation will be jointly dependent upon the increments of x and y, or the function will become

$$u_2=f(x+h,\ y+k).$$

In order to obtain the developement of this function, we may consider, first, the variation as arising from the increment of x, supposing y constant; and, secondly, from the increment of y, supposing x constant.

We may obtain the developement on each of these suppositions separately by Taylor's Theorem, in the following manner:

First, changing x into $x+h$ we shall have

$$f(x+h,y)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}.\frac{h^2}{1.2}+\&c.$$

in which series y in contained only in the functions

$$u,\ \frac{du}{dx},\ \frac{d^2u}{dx^2},\ \&c.$$

When therefore we change y into $y+k$ we must develope each of these functions u, $\frac{du}{dx}$, &c. for the increment $y+k$, supposing x constant. Thus we must replace, in the equation above,

$$u \text{ by } u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \&c.;$$

$$\frac{du}{dx} \text{ by } \frac{du}{dy} + \frac{d.\frac{du}{dx}}{dy}k + \frac{d^2\frac{du}{dx}}{dy^2}\cdot\frac{k^2}{1.2} + \frac{d^3\frac{du}{dx}}{dy^3}\frac{k^3}{2.3} + \&c.;$$

$$\frac{d^2u}{dx^2} \text{ by } \frac{d^2u}{dx^2} + \frac{d.\frac{d^2u}{dx^2}}{dy}k + \frac{d^2.\frac{d^2u}{dx^2}}{dy^2}\frac{k^2}{2} + \frac{d^3.\frac{d^2u}{dx^2}}{dy^3}\frac{k^3}{2.3} + \&c.;$$

&c., &c., &c., &c., &c.;

and forming as many lines as there are terms in the equation, we shall obtain

$$f(x+h, y+k) = \left.\begin{array}{l} u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{2} + \&c. \\ \quad + \frac{du}{dx}h + \frac{d.\frac{du}{dx}}{dy}hk + \&c. \\ \qquad + \frac{d^2u}{dx^2}\frac{h^2}{2} + \&c. \\ \qquad\qquad + \&c. \end{array}\right\}\cdots\cdots(A)$$

If we had made these substitutions in an inverse order, we should have found, first, by changing y into $y+k$,

$$f(x, y+k) = u + \frac{du}{dy}k + \frac{d^2u}{dy^2}\frac{k^2}{2} + \frac{d^3u}{dy^3}\frac{k^3}{2.3} + \&c.;$$

and putting then in each term $x+h$ in place of x, we should have arrived at this developement,

$$f(y+k, x+h) = \left.\begin{array}{l} u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{2} + \&c. \\ \quad + \frac{du}{dy}k + \frac{d.\frac{du}{dy}}{dx}hk + \&c. \\ \qquad + \frac{d^2u}{dy^2}\frac{k^2}{2} + \&c. \\ \qquad\qquad + \&c. \end{array}\right\}\cdots\cdots(B)$$

The order in which we make these substitutions being arbitrary, (since in putting $x+h$ wherever x enters, and $y+k$ wherever y enters, these operations cannot affect each other), it follows, that the two developements (A) and (B) must be identical, and consequently the terms affected by the same products of h and k have the same values.

Equating therefore the terms which are multiplied by hk, we shall obtain

$$\frac{d.\frac{du}{dx}}{dy}=\frac{d.\frac{du}{dy}}{dx}, \text{ or } \frac{d^2u}{dxdy}=\frac{d^2u}{dydx}; \quad \ldots\ldots (19)$$

an equation which shews us that, in taking the second differential of the product of two variables, the order of the differentiations is arbitrary.

The same thing may be proved for the differential coefficients of higher orders, by equating the differential coefficients of the other terms of the equations (A) and (B); which are multiplied into the same powers of h and k. Thus we have in general,

$$\frac{d^{n+m}u}{dy^n\,dx^m}=\frac{d^{m+n}u}{dx^m\,dy^n} \quad \ldots\ldots (20)$$

If we extend the definition of the differential of a function of one variable to those of two variables, we may derive the form of this differential from the preceding developement: for we have the increment of the function

$$f(x+h,y+k)-f(x,y)=\left\{\begin{array}{l}\frac{du}{dx}h+\&\text{c.}\\ +\frac{du}{dy}k+\&\text{c.}\end{array}\right.$$

To express the ratio of this increment relatively to those of the variables, we must divide the first series by h and the second by k: and in passing from the differential coefficient to the differential, we must multiply the first term by dx, and the second by dy: thus we have

$$df(x,y)=\frac{du}{dx}dx+\frac{du}{dy}dy \ldots\ldots (21).$$

It must be observed, that these terms mean respectively the differentiation of the function, with respect to one variable, the other being supposed constant. And taken separately they are called the *partial differentials* of the function: their *sum* constitutes the *total differential.*

For the further investigation of the differentials of functions of two variables, and their maxima and minima, the student is referred to Lacroix, (Transl.) p. 145, 164; Jephson's Fluxional Calculus, p. 140, &c.; or Boucharlat, (Transl.) p. 116.

APPLICATIONS
OF THE
DIFFERENTIAL CALCULUS.

Applications of the theory of the developement of functions to several cases of Algebraic and Transcendental functions.

ALGEBRAIC FUNCTIONS.

1. Let the function to be developed be,

$$y=\frac{1}{a+x};$$

differentiating, we find

$$dy=-\frac{dx}{(a+x)^2},$$

whence $\frac{dy}{dx}=-\frac{1}{(a+x)^2};$

and differentiating anew, we shall get successively

$$\frac{d^2y}{dx^2}=\frac{2(a+x)}{(a+x)^4}=\frac{2}{(a+x)^3},$$

$$\frac{d^3y}{dx^3}=-\frac{2.3(a+x)^2}{(a+x)^6}=-\frac{2.3}{(a+x)^4}.$$

&c.

Making then $x=0$ in the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, we find

$$(y)=\frac{1}{a},\ \left(\frac{dy}{dx}\right)=-\frac{1}{a^2},\ \left(\frac{d^2y}{dx^2}\right)=\frac{2}{a^3},\ \left(\frac{d^3y}{dx^3}\right)=-\frac{2\,.\,3}{a^4};$$

and substituting these values and that of y in the formula, we shall obtain

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&c.$$

2d. For a second application, take $y=\sqrt{a^2+bx}$; we have then

$$y=(a^2+bx)^{\frac{1}{2}},$$

$$\frac{dy}{dx}=\tfrac{1}{2}\,(a^2+bx)^{-\frac{1}{2}}\,b=\frac{1}{2}\,.\frac{b}{\sqrt{a^2+bx}}$$

$$\frac{d^2y}{dx^2}=-\tfrac{1}{2}\,.\,\tfrac{1}{2}\,(a^2+bx)^{-\frac{3}{2}}\,b^2=-\frac{\frac{1}{2}\,.\,\frac{1}{2}\,.\,b^2}{\sqrt{(a^2+bx)^3}},$$

$$\frac{d^3y}{dx^3}=\tfrac{1}{2}\,.\,\tfrac{1}{2}\,.\,\tfrac{3}{2}\,(a^2+bx)^{-\frac{5}{2}}\,b^3=\frac{\frac{1}{2}\,.\,\frac{1}{2}\,.\,\frac{3}{2}\,.\,b^3}{\sqrt{(a^2+bx)^5}};$$

making $x=0$, these values will become

$$(y)=(a^2)^{\frac{1}{2}}=a,\ \left(\frac{dy}{dx}\right)=\tfrac{1}{2}\,.\,\frac{b}{a},\ \left(\frac{d^2y}{dx^2}\right)=-\tfrac{1}{2}\,.\,\tfrac{1}{2}\,.\,\frac{b^2}{a^3},$$

$$\left(\frac{d^3y}{dx^3}\right)=\frac{\frac{1}{2}\,.\,\frac{1}{2}\,.\,\frac{3}{2}\;b^3}{a^5};$$

and substituting these values of (y), $\left(\frac{dy}{dx}\right)$ &c. in the formula, we shall find

$$\sqrt{a^2+bx}=a+\frac{bx}{2a}-\frac{b^2x^2}{8a^3}+\frac{b^3x^3}{16a^5}-\&c.$$

3d. By the same method we may obtain the developement of $y=(a+x)^m$, or in other words, we may prove the Binomial theorem in its most general form; since we have before established the differentiation of all powers whether whole or fractional, positive or negative, without assuming it. We have then

$$y=(a+x)^m,$$

$$\frac{dy}{dx}=m(a+x)^{m-1},$$

$$\frac{d^2y}{dx^2}=m(m-1)(a+x)^{m-2},$$

$$\frac{d^3y}{dx^3}=m(m-1)(m-2)(a+x)^{m-3}; \text{ \&c.}$$

making $x=0$, the value of y is reduced to a^m, whence $(y)=a^m$ and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, &c. give us

$\left(\frac{dy}{dx}\right)=ma^{m-1}$, $\left(\frac{d^2y}{dx^2}\right)=m(m-1)a^{m-2}$,

$\left(\frac{d^3y}{dx^3}\right)=m(m-1)(m-2)a^{m-3}$, which values of (y), $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, &c. being substituted in the formula, we find

$$(a+x)^m=a^m+ma^{m-1}x+m\frac{(m-1)}{2}a^{m-2}x^2$$
$$+m\frac{(m-1)}{2}\cdot\frac{(m-2)}{3}a^{m-3}x^3+\text{\&c.} \quad . \; . \; . \; (22)$$

In the foregoing instances we have made use of Stirling's series; in the following case we have an application of Taylor's theorem.

Let $y_2=\sqrt{x+h}$ be the function;
we have then

$$y=\sqrt{x}=x^{\frac{1}{2}},$$

and therefore

$$\frac{dy}{dx}=\tfrac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}},$$

$$\frac{d^2y}{dx^2}=-\tfrac{1}{4}x^{-\frac{3}{2}}=-\tfrac{1}{4}\cdot\frac{1}{x^{\frac{3}{2}}},$$

$$\frac{d^3y}{dx^3}=\tfrac{3}{8}x^{-\frac{5}{2}}=\tfrac{3}{8}\frac{1}{x^{\frac{5}{2}}},\ \&c.$$

substituting in the formula we have

$$\sqrt{x+h}=\sqrt{x}+\tfrac{1}{2}\frac{h}{\sqrt{x}}-\tfrac{1}{8}\frac{h^2}{\sqrt{x^3}}+\tfrac{1}{16}\frac{h^3}{\sqrt{x^5}},\ \&c.$$

TRIGONOMETRICAL FUNCTIONS.

Let $y_2=\sin.(x+h)$, whence it follows that $y=\sin. x$; and we may therefore form the successive differential coefficients thus;

$$\frac{dy}{dx}=\cos. x;\frac{d^2y}{dx^2}=-\sin. x;\frac{d^3y}{dx^3}=-\cos. x;$$

$$\frac{d^4y}{dx^4}=\sin. x;\frac{d^5y}{dx^5}=\cos. x;\ \&c.;$$

and substituting in Taylor's formula, we find

$$\sin.(x+h)=\sin. x+\cos. x.\frac{h}{1}-\sin. x\frac{h^2}{1.2}-\cos. x.\frac{h^3}{1.2.3}$$
$$+\sin. x.\frac{h^4}{1.2.3.4}+\cos. x\frac{h^5}{2.3.4.5},\ \&c.$$

Making $x=0$, $\sin. x$ will then be $=0$, $\cos. x=1$, and the series will become

$$\sin. h=h-\frac{h^3}{1.2.3}+\frac{h^5}{1.2.3.4.5}-\&c. \quad .\ .\ .\ .\quad (23)$$

If we took $y_2 = \cos.(x+h)$, we should find, proceeding as in the last example, that

$$\cos. h = 1 - \frac{h^2}{2} + \frac{h^4}{2.3.4} - \&c. \quad \ldots\ldots \quad (24)$$

LOGARITHMS.

In differentiating the exponential function, we found

$$\frac{dy}{dx} = Aa^x.$$

By means of an application of Stirling's theorem we may determine the value of A. Investigating by that theorem the developement of a^x, we have,

$$y = a^x,$$
$$\frac{dy}{dx} = Aa^x,$$
$$\frac{d^2y}{dx^2} = \frac{Ad.a^x}{dx} = \frac{A^2a^xdx}{dx} = A^2a^x,$$
$$\frac{d^3y}{dx^3} \ldots\ldots\ldots\ldots = A^3a^x, \&c.$$

and making $x=0$, we shall find

$$(y) = a^0 = 1, \left(\frac{dy}{dx}\right) = A, \frac{d^2y}{dx^2} = A^2, \left(\frac{d^3y}{dx^3}\right) = A^3, \&c.$$

Substituting these values in the series, we have

$$a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.; \quad \ldots\ldots \quad (25)$$

making $x = \frac{1}{A}$, this equation will become

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c.;$$

and representing by e, the second side of this equation, it will be changed into $a^{\frac{1}{A}}=e$, whence we find $a=e^A$; and taking the logarithms, we have

$$\log. a=\log. e^A=A \log. e;$$

therefore

$$A=\frac{\log. a}{\log. e}$$

Thus the value of A belonging to any system whose base is a, is determined by taking this ratio of the logarithms of a and e.

If, then, A belong to the system whose base is e, we have $A=\frac{\log. e}{\log. e}=1$.

The number e, whose value is given by the equation $e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+$ &c., is the base which Napier selected for calculating his tables of logarithms.

The series $1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+$ &c. being sufficiently convergent, we may take the first ten terms as an approximation, when we shall find the value of e to be 2,7182818 very nearly.

Thus e may be considered as that value of a which gives $A=1$.

In general, $\frac{1}{A}$ is called the *modulus* of a system whose base is a.

If we designate a logarithm in the Napierian system by the symbol λ, and one in any other system by l, we have as above

$$a^{\frac{1}{A}}=e;$$

or to base a, $le=\frac{1}{A}$

and $a=e^A$;

or to base e, $\lambda a = A$;
or the logarithm of the Napierian base, in any system, is the modulus of that system. And the Napierian logarithm of the base of any system is the reciprocal of its modulus.

The Napierian logarithms are often called *hyperbolic;* and sometimes *natural* logarithms. The modulus, being unity, they are most commonly used in analytics for the sake of simplicity.

In the Napierian system we have,

$$y = e^x = e^{\lambda y}.$$

Or taking the logarithms of these quantities in any system,

$$\begin{aligned} ly &= l\,(e^{\lambda y}) \\ &= \lambda y\,.\,le. \end{aligned}$$

Hence for the same quantity y, λy being constant, and le equal to the modulus of the assumed system, *the logarithms of* y, *in different systems, will be as the moduli of those systems.*

Thus if we have given the logarithm of a number in one system, we pass to its logarithm in another by multiplying by the modulus.

If we have $y = lx$,
and $y_2 = l\,(x + h)$,
we may develope y_2, in a series, by Taylor's formula. Taking the successive differentials, we have

$$dy = d\,.\,lx = \frac{dx}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}.$$

Whence
$$\frac{d^2y}{dx^2} = -\frac{1}{x^2},$$

$$\frac{d^3y}{dx^3} = \frac{2x}{x^4} = \frac{2}{x^3}.$$

And substituting these values,

$$l(x+h)=lx+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\&c. \dots\dots\dots \quad (26)$$

This series is not sufficiently convergent to be employed in the calculation of logarithms of numbers. We may however easily obtain a form which shall be sufficiently convergent.

If we had taken $x-h$, the form would become,

$$l\,(x-h)=lx-\frac{h}{x}-\frac{h^2}{2x^2}-\frac{h^3}{3x^3}-\&c.$$

Subtracting this equation from the former, we have,

$$l\,(x+h)-l\,(x-h)=2\left(\frac{h}{x}+\frac{h^3}{3x^3}+\&c.\right)$$

which, by the property of logarithms, is,

$$l\left(\frac{x+h}{x-h}\right)=2\left(\frac{h}{x}+\frac{h^3}{3x^3}+\right) \dots\dots\dots\dots \quad (27)$$

In this form, substituting particular numbers for $x+h$ and $x-h$, we may obtain some of the most fundamental logarithms, from which the others may be derived; and by various artifices in particular cases the processes may be shortened.

Other forms, more convergent, may also be obtained: for which, and the methods of calculating logarithms in general, see Boucharlat, Transl. p. 191, or Jephson's Calculus, p. 49, and the Introduction to Dr. Hutton's Tables.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE THEORY OF CURVES.

Many individual properties of curves are established by geometry; but we do not obtain any common principle by which they are connected.

The algebraic theory of curves gives us a more general view of them, connecting them and their fundamental properties together by one great principle, their reference to coordinates by equations.

But this theory does not enable us to discover all their properties. It shews us the limits within which a curve is included, and the general character of its form or branches.

We have still to inquire into many other particulars of its form, and its properties, referring to lines connected with it.

The differential calculus affords the means of investigating these points upon simple principles, connecting them all by a general theory.

ON TANGENTS, NORMALS, &c. TO CURVES.

Let the equation to any curve be

$$y=f(x).$$

Let there be taken two ordinates, y and y_2, corresponding to the abscissæ x, $x+h$. Draw a secant through their extremities, meeting the axis in S, (fig. 1.)

and through *P* draw a parallel to the axis, which will intercept a part of the second ordinate $= y_2 - y$, and will be itself $= h$. Then we have, by similar triangles,

$$\frac{y_2 - y}{h} = \frac{y}{SM} \therefore SM = y . \frac{h}{y_2 - y}.$$

But y_2 may be developed by Taylor's theorem, or we have

$$y_2 = f(x+h) = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \&c.$$

Hence subtracting

$$y_2 - y = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \&c.$$

Substituting this value in that of *SM*, we shall find

$$SM = \frac{hy}{\frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \&c.}$$

and dividing by *h*,

$$SM = \frac{y}{\frac{dy}{dx} + \frac{d^2y}{dx^2} \cdot \frac{h}{1.2} + \&c.}$$

At the limit, $h = 0$, and *SM* becomes *MT*, which gives us

the subtangent, $MT = \frac{y}{\frac{dy}{dx}}, = y \frac{dx}{dy}$, ... (28)

Drawing a perpendicular to the tangent at *P* (fig. 2, 3) and the ordinate, we have by similar triangles,

$$\frac{MT}{y} = \frac{y}{MN};$$

or substituting for MT

$$\frac{dx}{dy}=\frac{y}{MN}; \text{ hence we have}$$

the subnormal $MN=y\,.\,\dfrac{dy}{dx}\;.\;.\;.\;.\;.\;(29).$

In respect to the tangent and the normal, we have

$$PT=\sqrt{\overline{MT}^2+\overline{PM}^2},$$

$$\text{or } \textit{tangent}=\sqrt{y^2\frac{dx^2}{dy^2}+y^2}=y\sqrt{\frac{dx^2}{dy^2}+1}\,;\;.\;.\;.\;.\;(30)$$

$$PN=\sqrt{\overline{MN}^2+\overline{PM}^2},$$

$$\text{or } \textit{normal}=\sqrt{y^2\frac{dy^2}{dx^2}+y^2}=y\sqrt{\frac{dy^2}{dx^2}+1}.\;.\;.\;.\;.\;(31)$$

To find the equation to the tangent, let x_2 and y_2 be the coordinates of the point of contact P; the equation to the straight line PT, passing through the point P, may then be represented by

$$y-y_2=A(x-x_2),$$

where A is the trigonometrical tangent of the angle MTP, which the tangent at P forms with the axis, and will therefore be expressed by $\dfrac{PM}{MT}$; therefore (writing the angle $MTP=\phi$)

$$\tan.\ \phi=\frac{PM}{MT}=\frac{y_2}{\textit{subtangent}}=\frac{y_2}{y^2\dfrac{dx_2}{dy_2}}=\frac{dy_2}{dx_2}\;.\;.\;.\;.\;.\;(32)$$

Substituting this value of A in the *equation to the tangent,* that equation becomes

$$y-y_2=\frac{dy_2}{dx_2}(x-x_2). \quad \ldots\ldots (33)$$

The *equation to the normal* will therefore be

$$y-y_2=-\frac{dx_2}{dy_2}(x-x_2). \quad \ldots\ldots (34)$$

ON ASYMPTOTES TO CURVES.

We have already found in all cases

$$\frac{dy}{dx}=\tan.\ \phi.$$

If then in this expression, deduced for any particular curve, we *suppose* x, *or* y, *or both*, (according to the conditions of the curve,) *to become infinite*, we shall have the *inclination* of the *tangent belonging to a point infinitely remote*, or that of the asymptote.

To find its *position*, or the point at which it meets the axis, we must take the value of the subtangent — the abscissa, (or the distance from the vertex to where the tangent meets the axis,) and substitute x, or y, or both, $=\infty$. If the expression continue finite, by this value, and that of tan. ϕ, we have the *position* of *the asymptote*, and can consequently construct it geometrically.

MAXIMA AND MINIMA OF THE ORDINATE AND ABSCISSA OF CURVES.

Since $y=f(x)$ may represent the equation to any curve, the general principles of the investigation of maxima and minima apply to the maximum or minimum of the ordinate. From the given equation to the curve we determine whether $\frac{dy}{dx}$ be $= 0$; and also whether the second differential coefficient be $=0$ or not, and so successively, till we obtain one which does not vanish, whose sign determines whether it be a maximum or a minimum.

Some curves admit of neither; some have both; some only one or the other.

By the formula (32) $\tan. \phi = \frac{dy}{dx}$.

But when the ordinate of the point of contact is either a maximum or a minimum, we have just seen that we have

$$\frac{dy}{dx}=0$$

$$\therefore \tan. \phi = 0;$$

that is, the *tangent* will in either case be *parallel to the axis X*, (as in fig. 6, 7.)

Since it is immaterial which of the coordinates we designate as x and which as y, we may write the equation

$$x=f(y)$$

and the condition

$$\frac{dx}{dy}=0$$

will give the maximum or minimum of the abscissa.

But we hence also deduce

$$\frac{dy}{dx}=\frac{1}{0}=\infty.$$

Therefore keeping to the original form of the equation, this last condition gives the maximum or minimum of the abscissa.

Also this condition, according to what was just observed, indicates that the tangent is *perpendicular* to the axis of x.

SINGULAR POINTS OF CURVES.

POINTS OF INFLEXION.

The determination of the maximum or minimum of the ordinate or abscissa gives us the limits within which the curve is included: but we have still to trace the particular form of its course within those limits. We shall therefore proceed to investigate the mode of determining whether the curve be *concave* or *convex* towards the axis: also, if at any point in its course it *change from concave to convex;* which is called a *point of inflexion:* and again, whether its *branches* at any point or points *touch or intersect one another.* The points in a curve at which any of these conditions take place are called *singular points;* and the differential calculus affords us the means of determining them, so as to give us the complete description of the curve.

(Fig. 4, 5.) Let there be drawn a tangent to any curve at a point P whose coordinates are x, y: take another ordinate $y+k$ corresponding to $x+h$: which being produced to meet the tangent, let the increment limited by the tangent be l: and the angle at which the tangent meets the axis, or a parallel to it through P, be ϕ. Then from the right-angled triangle we have

$$l = h. \tan. \phi,$$

which by (32) $= h \frac{dy}{dx}$.

Hence we have

$$k - l = f(x+h) - y - \frac{dy}{dx} \cdot h$$

$$= \left(y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \&c.\right) - y - \frac{dy}{dx} \cdot h$$

$$= \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \&c. \ldots\ldots (35)$$

We have here supposed $k > l$. If we had $k < l$, we should take

$$l - k = \frac{dy}{dx} h - \left(f(x+h) - y\right)$$

$$= -\frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} - \&c.$$

or the sign of the second differential coefficient will be,

+, when $k > l$,

That is, when the curve is *convex* to the axis:

−, when $k < l$,

That is, when the curve is *concave* to the axis.

On the other side of the axis, or with $-y$, the same investigation holds good; but the signs will be reversed. Hence we may say, generally, that on whatever side the curve falls, $\frac{d^2y}{dx^2}$ *has the same sign as* y when the curve is *convex* to the axis of the abscissæ, and has a *contrary sign* when the curve is *concave* to the same axis.

The curve being convex or concave to the axis of the abscissæ accordingly as the ordinate is arrived at its minimum or its maximum, we see the reason why $\frac{d^2y}{dx^2}$ is positive in the first case and negative in the second.

In this investigation we have taken the ordinate corresponding to $x+h$: let us now take it at any equal distance on the other side of P, or corresponding to $x-h$, (fig. 8). In this case, changing h into $-h$ in the expression for $k-l$, we shall have

$$k_2-l_2=f(x-h)-y+\frac{dy}{dx}h.$$

Replacing now the expressions $f(x+h)$ and $f(x-h)$ by their developements, we shall have

$$k-l=\left(y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.\right)$$
$$-y-\frac{dy}{dx}h,$$
$$k_2-l_2=\left(y-\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.\right)$$
$$-y+\frac{dy}{dx}h;$$

and by reducing, these equations will become

$$\left.\begin{aligned} k-l&=\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c. \\ k_2-l_2&=\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c. \end{aligned}\right\}\;\cdots\cdots\;(36)$$

In order now that there may be a point of inflexion at P, it is necessary that when we give to h an exceedingly small value, the lines $k-l$ and k_2-l_2 should fall one above and the other below the tangent, and that consequently $k-l$ and k_2-l_2 should have different signs. But this is not possible, unless *the first term in each of the series be* 0; for if that term be not 0, then we may give to h a value so small that the term $\frac{d^2y}{dx^2}\frac{h^2}{1.2}$ shall be greater than the sum of all the terms following, and therefore the sign of that term be the sign of the whole series; and since that term is the same in the two series, it follows that $k-l$ and k_2-l_2 would in this case have the same sign: *in order, therefore, that they may have different signs, we must have*

$$\frac{d^2y}{dx^2}\frac{h^2}{1.2}=0, \text{ or } \frac{d^2y}{dx^2}=0.$$

If it should happen that the same value of x, which makes $\frac{d^2y}{dx^2}$ vanish, make also $\frac{d^3y}{dx^3}$ vanish, then, in order that there may be a point of inflexion, $\frac{d^4y}{dx^4}$ must become 0 likewise; and if in this case $\frac{d^5y}{dx^5}$ result 0, $\frac{d^6y}{dx^6}$ must result 0 also: *and generally the last differential coefficient that vanishes must be of an even order.*

If the value of x, which is the same in the developements, be such that $\frac{d^2y}{dx^2}$ become infinite, the two developements will be so likewise; and we can then conclude nothing from the preceding demonstration, which rests on the supposition of these developements being possible. In this case we must observe that the condition $\frac{d^2y}{dx^2}=0$ indicates, generally, that $\frac{d^2y}{dx^2}$ ought to change its sign at the point of inflexion, which agrees with what was proved before: but this change of sign may also take place in passing through infinity; for let

$$\frac{d^2y}{dx^2}=\frac{b^2}{x-a};$$

if, then, we substitute successively for x the values

$$\begin{aligned} x&=a-h, \text{ we shall find} & \frac{d^2y}{dx^2}&=-\frac{b^2}{h}, \\ x&=a & \frac{d^2y}{dx^2}&=\infty, \\ x&=a+h, & \frac{d^2y}{dx^2}&=+\frac{b^2}{h}, \end{aligned}$$

where we see that it is the denominator of the value of $\frac{d^2y}{dx^2}$, which produces the change of sign in the differential coefficient, after passing the point of inflexion.

Hence it follows, that if there be a point of inflexion in a curve, we must have, for the abscissa of that point,

$$\frac{d^2y}{dx^2}=0; \text{ or } \frac{d^2y}{dx^2}=\infty.$$

When, therefore, we have ascertained that one of these conditions is fulfilled, we must successively augment and diminish the abscissa of the point which fulfils the condition by a very small quantity h; and if, for these new values of x, $\frac{d^2y}{dx^2}$ has different signs, we may then conclude that there is a point of inflexion: for when $\frac{d^2y}{dx^2}$ is positive, the curve is convex to the axis of the abscissæ, and concave to that axis when $\frac{d^2y}{dx^2}$ is negative; but it is by this change from convex to concave, or from concave to convex, that the curve manifests its point of inflexion.

MULTIPLE POINTS.

It may happen that the equation of a curve shall give $\frac{dy}{dx}$ with two or more unequal values, this will happen if the expression involve some root which may be connected with the other terms, either by the sign $+$ or $-$.

But since $\frac{dy}{dx} = \tan. \phi$, it follows that there are as many tangents to the curve at the point to which $\tan. \phi$ belongs, as there are values of the differential coefficient; that is, the curve must intersect itself so many times at that point; or it is called a *multiple point:* being a double, triple, &c. point, according as two, three, &c. branches intersect.

Since the differential coefficient has several values, it is expressed by writing

$$\frac{dy}{dx} = \frac{0}{0}.$$

By finding these values we discover at what angles the branches of the curve intersect the axis.

In some curves it is not difficult to investigate these values without using any other methods than those here explained; but the *general* investigation of such points cannot be effected without having recourse to more complex considerations relative to the values of the expression $\frac{0}{0}$, which are not of a nature to fall within the plan of so elementary a treatise as the present.

Fig. (11) represents a curve with a double, and fig. (12) one with a triple point.

The reader who is desirous of fully investigating these points is referred to Jephson's Calculus, chap. 5 and 12; to Boucharlat, (Trans.) p. 50 and 90; or Lardner's Diff. Calc. sect. 15.

POINTS OF REFLEXION, &c.

If the equation to a curve give only one value of $\frac{dy}{dx}$ there will be but one tangent at a given point; but if at the same time $\frac{d^2y}{dx^2}$ has several values, then it is evident from (35) that the developement of $k-l$ will have several values; or there will be *several branches* of the curve having a *common tangent.*

If these several values have all the same sign as y, the branches are all convex to the axis; if different, they lie opposite ways.

If this happen at a point where the curve terminates from values of x beyond it giving y impossible, the point of several contacts is called a *cusp,* or point of *reflexion:* it is said to be of the *first species,* when the curves are *opposed,* and of the second, when the concavities lie the *same way.*

The two species of cusps are represented in figs. (9, 10.) These points are easily determined in particular curves. For their connection with the general theory of the values of $\frac{0}{0}$ the reader is referred to the works already named.

In the theory of curves we have instances of *conjugate points,* which are in most cases easily determined by the construction of the equation; these points are also referred to the theory of the values of $\frac{dy}{dx}=\frac{0}{0}$; but for these investigations the reader must have recourse to the same works as in the former cases.

ON THE

OSCULATION OF CURVES;

CURVATURE, EVOLUTES, INVOLUTES, &c.

OSCULATING CURVES.

Let $y=\phi x$ and $y=Fx$ be the equations to two curves which meet (fig. 13.) in the point P whose co-ordinates are $x\ y$, we shall have then, for that point,

$$\phi x = Fx;$$

and supposing that x becomes $x+h$, the preceding equations will give the ordinates belonging to the same abscissa in the two curves,

$$\phi\ (x+h) = \phi x + \frac{d\,.\,\phi x}{dx} h + \frac{d^2 \phi x}{dx^2}\,\frac{h^2}{1.2} + \&c.$$

$$F(x+h) = Fx + \frac{d\,.\,Fx}{dx} h + \frac{d^2.Fx}{dx^2}\,\frac{h^2}{1.2} + \&c.$$

If now all the corresponding terms of these developements be identically the same, the two ordinates will have the same values, and the two curves will coincide; but if we have only $\phi x = Fx$, the curves, as we have just seen, will have merely a common point P; if, besides $\phi x = Fx$, we have

$$\frac{d.\phi x}{dx} = \frac{d.Fx}{dx},$$

the difference between the two ordinates belonging to the same abscissa will be less, and the curves will then approach more nearly to each other; and still more, if, in addition to these equations, we have also

$$\frac{d^2\phi x}{dx^2} = \frac{d^2Fx}{dx^2};$$

and so on in order; the difference becoming continually less, the greater the number of terms respectively equal in their developements.

This being premised, let a, b, c, &c., be the constants in the equation $y = Fx$; we may then, without changing the nature of the curve, give arbitrary values to these constants, so that we may have,

$$y = Fx, \; \frac{dFx}{dx} = \frac{d\phi x}{dx}, \; \frac{d^2Fx}{dx^2} = \frac{d^2\phi x}{dx^2}, \text{ \&c.},$$

or taking as many of these equations as there are constants, determine the constants by the condition that those equations are satisfied.

For example, if the equation $y = Fx$ contain three constants, a, b, c, we may put

$$Fx = \phi x, \; \frac{dFx}{dx} = \frac{d\phi x}{dx}, \; \frac{d^2Fx}{dx^2} = \frac{d^2\phi x}{dx^2},$$

deduce from these equations the values of a, b, c, in functions of x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and substitute them in the equation $y=Fx$; which will then possess this property, that when we put $x+h$ for x, *the developement for the first curve will have the three first terms respectively equal to the three first terms of the developement for the second curve.*

The same reasoning will apply to an equation containing any other number of constants.

Thus we have seen that if the curves $y=\phi x$ and $y=Fx$ have only a common point, and x, y, be the coordinates of that point, we shall have the equation of condition $Fx=\phi x$; but if we determine two constants of the equation $y=Fx$, by the conditions $Fx=\phi x$, and $\frac{dFx}{dx}=\frac{d\phi x}{dx}$, the curves will begin to approach each other.

DEF. Let $y=fx$ represent what $y=Fx$ becomes after we have substituted the values of these *two* constants; then $y=fx$ *is called an osculate of the first order to the curve* $y=\phi x$; and if (always by virtue of the arbitrary values that may be given to the constants) we have eliminated *three* of the constants of the equation $y=Fx$, by means of the following equations:

$$Fx=\phi x, \quad \frac{dFx}{dx}=\frac{d\phi x}{dx}, \quad \frac{d^2Fx}{dx^2}=\frac{d^2\phi x}{dx^2},$$

and $y=\psi x$ represent what $y=Fx$ becomes after this substitution, the curve $y=\psi x$ is called *an osculate of the second order to the curve* $y=\phi x$, which it will approach still nearer than $y=fx$ does, and so on, in order; so that for an *osculate of the* n*th order, we shall have the equations determining* n + 1 *constants,*

$$Fx=\phi x,\ \frac{dFx}{dx}=\frac{d\phi x}{dx},\ \frac{d^2Fx}{dx^2}=\frac{d^2\phi x}{dx^2}\ \ldots$$
$$\frac{d^nFx}{dx^n}=\frac{d^n\phi x}{dx^n}\ \ldots\ldots(37)$$

We will proceed to shew, that of *two osculates* which we have thus obtained, by giving arbitrary values to the constants of the same equation, *the osculate of an inferior order cannot pass between the other and the curve*, in respect to which the osculation takes place. For example, let B (fig. 13.) be the curve $y=\phi x$, and C its osculate $y=\psi x$ of the second order; we have then to demonstrate that the osculate D, or $y=fx$ of the first order cannot pass betwixt the curves B and C.

For this purpose, by putting $x+h$ in place of x, in these equations, we shall find

$$\phi(x+h)=\phi x+\frac{d\phi x}{dx}h+\frac{d^2\phi x}{dx^2}\frac{h^2}{1.2}+\frac{d^3\phi x}{dx^3}\frac{h^3}{2.3}+\&c.,$$
$$\psi(x+h)=\psi x+\frac{d\psi x}{dx}h+\frac{d^2\psi x}{dx^2}\frac{h^2}{1.2}+\frac{d^3\psi x}{dx^3}\frac{h^3}{2.3}+\&c.,$$
$$f(x+h)=fx+\frac{d.fx}{dx}h+\frac{d^2fx}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3fx}{dx^3}\cdot\frac{h^3}{2.3}+\&c.,$$

and $y=\psi x$ being an osculate of the second order to $y=\phi x$, we must have

$$\psi x=\phi x,\ \frac{d\psi x}{dx}=\frac{d\phi x}{dx},\ \frac{d^2\phi x}{dx^2}=\frac{d^2\phi x}{dx^2};$$

$y=fx$ being an osculate of the first order to $y=\phi x$, we must have also

$$fx=\phi x,\ \frac{dfx}{dx}=\frac{d\phi x}{dx};$$

from which equations we have therefore

$$\phi x = \psi x = fx,$$
$$\frac{d\phi x}{dx} = \frac{d\psi x}{dx} = \frac{dfx}{dx}$$

and only

$$\frac{d^2\phi x}{dx^2} = \frac{d^2\psi x}{dx^2}.$$

Let us then make

$$\phi x + \frac{d\phi x}{dx} h = K,$$
$$\tfrac{1}{2} \cdot \frac{d^2\phi x}{dx^2} = V;$$

and the three preceding developements may be written thus,

$$\phi\ (x+h) = K + Vh^2 + \frac{d^3\phi x}{dx^3} \frac{h^3}{2.3} + \&c.,$$
$$\psi\ (x+h) = K + Vh^2 + \frac{d^3\psi x}{dx^3} \cdot \frac{h^3}{2.3} + \&c.$$
$$f\ (x+h) = K + \frac{d^2fx}{dx^2} \frac{h^2}{1.2} + \frac{d^3fx}{dx^3} \cdot \frac{h^3}{1.2.3} + \&c.;$$

and observing that all the terms, commencing from that which is affected by h^3, have h^3 for a common factor, we may suppose

$$\frac{d^3\phi x}{dx^3} \frac{h^3}{2.3} + \&c. = Mh^3;$$

whence, making similar reductions in the other equations, we shall have

$$\phi\,(x+h) = K + Vh^2 + Mh^3,$$
$$\psi\,(x+h) = K + Vh^2 + Nh^3,$$
$$f\,(x+h) = K + \tfrac{1}{2} \frac{d^2fx}{dx^2} h^2 + Ph^3.$$

Now the curves $y = fx$ and $y = \psi x$ being osculating curves, one of the first and the other of the second order, V must necessarily differ from $\frac{1}{2}\frac{d^2fx}{dx^2}$; and we can therefore make only two hypotheses respecting V, viz.

$$V < \tfrac{1}{2}\frac{d^2fx}{dx^2}, \text{ or } V > \tfrac{1}{2}\frac{d^2fx}{dx^2}.$$

If V be less than $\frac{1}{2}\frac{d^2fx}{dx^2}$, let Z be the excess of $\frac{1}{2}\frac{d^2fx}{dx^2}$ over V, then we shall have

$$V + Z = \tfrac{1}{2}\frac{d^2fx}{dx^2},$$

where Z is a positive quantity; but if, on the contrary, V be greater than $\frac{1}{2}\frac{d^2fx}{dx^2}$, Z will be negative.

Substituting this value of $\frac{1}{2}\frac{d^2fx}{dx^2}$ in that of $f(x+h)$, and observing that h^2 is a common factor, our three developements will now become

$$\begin{aligned}\phi\,(x+h) &= K + (V + Mh)\,h^2,\\ \psi\,(x+h) &= K + (V + Nh)\,h^2,\\ f\,(x+h) &= K + (V + Z + Ph)\,h^2;\end{aligned}$$

and by making h exceedingly small, the quantity Z, which is independent of h, may become greater than the expressions Mh and Nh, which tend to 0.

In this case if Z be positive, $f(x+h) > \phi\,(x+h) > \psi\,(x+h)$, or the ordinate to D is greater than the ordinates to B and C; or D cannot pass between the other two curves.

If Z be negative, the ordinate to D is less than either of the ordinates to *B* and *C*, or *D* cannot pass between *B* and *C*.

ORDERS OF CONTACT.

If we take the case in which one of the equations is that of a *straight line*, or

$$y = Fx = ax + b,$$

the equations for the two constants (as before explained) will be,

$$\phi x_2 = ax_2 + b, \quad \frac{d\phi x_2}{dx_2} = a.$$

Where ϕx_2 belongs to the curve and may be replaced by y_2, hence,

$$y_2 = ax_2 + b, \quad \frac{dy_2}{dx_2} = a.$$

Whence by substitution,

$$y_2 = \frac{dy_2}{dx_2} x_2 + b;$$

or substituting for b

$$y_2 = \frac{dy_2}{dx_2} x_2 + y - \frac{dy_2}{dx_2} x.$$

Whence by transposition,

$$y - y_2 = \frac{dy_2}{dx_2} (x - x_2);$$

which is the equation to the tangent at $x_2 y_2$. Thus then, according to the definition, this osculate of the first order, is a tangent to the curve: also it follows from our theory, that between that straight line and the curve, we cannot draw any other straight line; which is a property of the tangent.

The tangent is said to have a *contact* of the first order with the curve; and generally an *osculate of the order n* has a *contact of the same order* with the curve to which it is an osculate; thus when we have, between the two curves, the equations

$$\phi x = Fx,\ \frac{d\phi x}{dx} = \frac{dFx}{dx},\ \frac{d^2\phi x}{dx^2} = \frac{d^2Fx}{dx^2},$$

these curves have with each other a contact of the second order; if, besides these equations, we have also

$$\frac{d^3\phi x}{dx^3} = \frac{d^3Fx}{dx^3},$$

the contact will be of the third order, and so on.

OSCULATING CIRCLE.

The equation to the circle, which is

$$(y-\beta)^2 + (x-\alpha)^2 = \gamma^2,$$

contains *three* constants; and we may therefore determine the circle which has a contact of the *second* order with any curve of which we know the equation. For this purpose, let x_2 and y_2 be the coordinates of the point P in the circumference of the circle; the value of y_2 will then be given by the equation

$$(y_2-\beta)^2 + (x_2-\alpha)^2 = \gamma^2,$$

and must replace Fx_2 in the equations of contact, which are

$$\phi x = Fx, \frac{d\phi x}{dx} = \frac{dFx}{dx}, \frac{d^2\phi x}{dx^2} = \frac{d^2Fx}{dx^2}.$$

If at the same time we take x and y for the coordinates of the curve $y = \phi x$, at the point of contact, the preceding equations will become

$$y = y_2, \frac{dy}{dx} = \frac{dy_2}{dx_2}, \frac{d^2y}{dx^2} = \frac{d^2y_2}{dx_2{}^2},$$

and in these we must substitute for the quantities y_2, $\frac{dy_2}{dx_2}$ and $\frac{d^2y_2}{dx_2{}^2}$, their values derived from the equation to the circle, and its successive differentials, which are

$$(y_2 - \beta)\frac{dy_2}{dx_2} + x_2 - \alpha = 0,$$

$$(y_2 - \beta)\frac{d^2y_2}{dx_2{}^2} + \frac{dy_2{}^2}{dx_2{}^2} + 1 = 0.$$

Or substituting in these equations, (which since when $y = y_2$, we have $x = x_2$, is done by suppressing the accents,) we have

$$(y - \beta)^2 + (x - \alpha)^2 = \gamma^2. \; \ldots\ldots \; (38)$$

$$(y - \beta)\frac{dy}{dx} + x - \alpha = 0. \; \ldots\ldots \; (39)$$

$$(y - \beta)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} + 1 = 0. \; \ldots\ldots \; (40)$$

from the last of which equations we deduce

$$y - \beta = -\frac{\left(1 + \frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}},$$

and putting this value in the equation (39) we obtain

$$x-\alpha=\frac{\left(1+\frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}}\frac{dy}{dx}.$$

If in the equation (38), we substitute these values of $y-\beta$ and $x-\alpha$, it becomes

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left(1+\frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}\frac{dy^2}{dx^2}=\gamma^2,$$

and, adding the numerators which have a common factor, we shall have

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^2\left(1+\frac{dy^2}{dx^2}\right)}{\left(\frac{d^2y}{dx^2}\right)^2}=\gamma^2;$$

an equation which reduces itself to

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2}=\gamma^2,$$

and, by extracting the square root, gives

$$\pm\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\gamma.$$

The double sign refers to the position of γ: if the curve be concave to the axis of x, then $\frac{d^2y}{dx^2}$ will be negative; and in order that γ may then result positively, we must take γ with the negative sign, and write

$$\gamma = -\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \ldots\ldots\ldots\ldots \quad (41)$$

for the curve being concave to the axis of the abscissæ $\frac{d^2y}{dx^2}$ appears as a negative quantity, and therefore, when substituted in the equation, will render the value of γ positive.

If the curve ought to be convex to the axis of x, the positive sign must be prefixed to the value of γ.

The value of γ is sometimes expressed in this form

$$\gamma = -\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\ d^2y}; \quad \ldots\ldots\ldots \quad (42)$$

a form which is readily deduced from the equation (41); for by reducing the two terms within the brackets to a common denominator, and observing that $(dx^2)^{\frac{3}{2}}$ is dx^3 we obtain

$$\gamma = -\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx^3.\frac{d^2y}{dx^2}} = -\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\ d^2y}.$$

DEF. The circle which osculates with a given curve at any point is called the *circle of curvature* for that point, and its radius, the *radius of curvature;* this then is easily obtained from the equation of the curve by taking its differential coefficients, and substituting in the above formula.

The osculating circle has received the name of the circle of curvature, because by means of it we may compare or measure the curvature of the curve at any point; for if at that point we describe, with the radius of curvature, an exceedingly small circular arc, that arc may be considered as the arc of the curve itself, from which it separates but in a very slight degree. Now the greater be the curvature of the circular arc, the less is its radius; and it follows, therefore, that from the decrease or increase of the radius of curvature, we may determine the increase or decrease of the curvature of the curve.

EVOLUTES AND INVOLUTES.

Since $\frac{dy}{dx}$ expresses the trigonometrical tangent of the angle which the tangent at P makes with the axis of x, the equation of the normal to a point whose coordinates are α and β, will be,

$$y-\beta=-\frac{dx}{dy}(x-\alpha);$$

and this equation being the same with the equation (39.) in which α and β are the coordinates of the centre of the osculating circle, we see that *the radius of that circle coincides in position with the normal to the curve.*

Def. If now, through all the points of a curve P, P_2, P_3, &c. (fig. 15.), we draw the radii of curvature PO, P_2O_2, P_3O_3, &c., we shall construct a series of points O, O_2, O_3, &c.; which points being all subject to a certain law, their locus constitutes a new curve dependent on the former, and which we call the *evolute* of the curve *P*, this latter curve, considered relatively to the evolute, being called the *involute*.

If now we pass from one point of the evolute to another, not only will x and y vary, but α, β, and γ will also vary at the same time; for since α and β are, generally, the coordinates of the centre of the osculating circle, and the evolute is formed by the system of those centres, it follows that α and β are the coordinates of the evolute, and therefore coordinates which must vary for different points of the curve. It is the same with γ, which is the radius of the osculating circle, and represents the distance of any point of the evolute from that point of the involute whence γ is drawn; and consequently, by differentiating the equation (39), in respect of all the letters, and dividing by dx, we shall obtain

$$(y-\beta)\frac{d^2y}{dx^2}+\frac{dy^2}{dx^2}-\frac{dy}{dx}\cdot\frac{d\beta}{dx}+1-\frac{d\alpha}{dx}=0;$$

and subtracting the equation (40) from this, there remains

$$-\frac{dy}{dx}\cdot\frac{d\beta}{dx}-\frac{d\alpha}{dx}=0,$$

whence we find

$$\frac{dy}{dx}=-\frac{\frac{d\alpha}{dx}}{\frac{d\beta}{dx}}=-\frac{d\alpha}{d\beta}$$

which value of $\frac{dy}{dx}$ being substituted in the equation (39), we shall obtain by transposing,

$$y-\beta=\frac{d\beta}{d\alpha}(x-\alpha)\ .\ .\ .\ .\ .\ .\ .\ (43).$$

We saw, above, that the equation

$$y-\beta=-\frac{dx}{dy}(x-\alpha)$$

was that of the radius of curvature, passing through the point whose coordinates are x and y; and it will therefore be always the equation of the same radius, when $-\frac{dx}{dy}$ is replaced by $\frac{d\beta}{d\alpha}$. But the equation (43) is also that of a *tangent* drawn at the point of the *evolute*, whose coordinates are α and β; and the *radius of curvature is therefore a tangent to the evolute.*

Since in the following demonstration we shall have to employ the differential of an arc of a curve, we will proceed to find that differential.

The arc of a curve (s), is a function of x and y; and y being a function of x, the arc is a function of x.

The increment of the abscissa, h, that of the ordinate k, and the chord c of the increment of the arc, form a right angled triangle; whence we have

$$c^2=h^2+k^2.$$

But $k=f(x+h)-fx=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.;$

substituting which value squared in the expression for c^2, and representing the coefficients of h^3, h^4, &c. by A, B, &c. we shall have

$$c^2=h^2+\frac{dy^2}{dx^2}h^2+Ah^3+Bh^4+\&c.,$$

$$\therefore\frac{c^2}{h^2}=\left(1+\frac{dy^2}{dx^2}\right)+Ah+Bh^2+\&c.,$$

$$\therefore\frac{c}{h}=\sqrt{1+\frac{dy^2}{dx^2}+Ah+Bh^2+\&c.}$$

When $h=0$, $c=0$. Hence we have

$$\frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}};$$

whence, multiplying by dx, we deduce

$$ds=\sqrt{dx^2+dy^2}.$$

For the evolute the coordinates are α and β; and for it, therefore, we shall have in like manner

$$ds=\sqrt{d\alpha^2+d\beta^2}.$$

If now we differentiate the equation (38) in respect of all the letters, we shall find

$$(y-\beta)(dy-d\beta)+(x-\alpha)(dx-d\alpha)=\gamma d\gamma,$$

and equation (39) gives us

$$(y-\beta)\,dy+(x-\alpha)\,dx=0,$$

which equation being subtracted from the preceding one, we have remaining

$$-(y-\beta)\,d\beta-(x-\alpha)\,d\alpha=\gamma d\gamma \quad \ldots\ldots\ldots \quad (44).$$

Substituting in this equation, and in the equation (38) the value of $y-\beta$, given by the equation (43), we shall find the two equations

$$-\frac{d\beta^2}{d\alpha}(x-\alpha)-(x-\alpha)\,d\alpha=\gamma d\gamma,$$

$$\frac{d\beta^2}{d\alpha^2}(x-\alpha)^2+(x-\alpha)^2=\gamma^2;$$

which, putting $x-\alpha$ as a common factor, and extracting the square root of the second, become

$$-(x-\alpha)\,.\,\frac{d\beta^2+d\alpha^2}{d\alpha}=\gamma d\gamma,$$

$$(x-\alpha)\frac{\sqrt{d\alpha^2+d\beta^2}}{d\alpha}=\gamma,$$

and dividing the first of these equations by the second, we obtain

$$d\gamma = -\sqrt{d\beta^2 + d\alpha^2}.$$

But we have seen, that, representing by s an arc of the evolute, we have

$$ds = \sqrt{d\beta^2 + d\alpha^2};$$

comparing which equation with the preceding one, we deduce

$$d\gamma = -ds, \text{ or } d(\gamma + s) = 0; \quad \ldots\ldots\ldots \quad (45)$$

and since every function whose differential is 0 must be constant, we have $\gamma + s =$ a constant quantity. And therefore if the radius of curvature increase, the arc s must diminish, and by the same quantity; a relation which we express by saying that *the radius of curvature varies by the same differences as the arc of the evolute.*

Thus in fig. (15) $\gamma\ \gamma_2\ \gamma_3$, &c. being successive radii of curvature to the points $P\ P_2\ P_3$, &c. $O\ O_2\ O_3$, are the corresponding points in the evolute: and measuring from some point B the arcs BO, $BO_2 = s$, s_2, &c. we have, for all the values of γ,

$$\gamma + s = \gamma_2 + s_2 = \gamma_3 + s_3, \text{ \&c.}$$

$$\therefore \gamma_2 - \gamma = s - s_2, \text{ \&c.} \quad \ldots\ldots\ldots \quad (46)$$

or the *difference of two radii of curvature is equal to the arc comprehended between them.*

It follows from this, that if on the evolute OB (fig. 15) we wrap a string BOP, which, being of course a tangent to the evolute at the point O where it leaves it, has its extremity in the point P of the involute, when we unwrap this string, keeping it constantly stretched, its extremity P will trace out in its course the involute; for supposing that in the course of its motion it has reached the position O_2P_2, it will be increased by OO_2, and will consequently be equal in length to the radius of curvature passing through the point O_2, whence the extremity P_2 of the string will be still in the involute.

The equation of the evolute is determined in the following manner: 1st. we deduce from the equation of the curve, the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$; 2d. we substitute these values in the equations (39) and (40), when we shall get two new equations, which are functions of x alone; 3d. eliminating x between these equations, we arrive at an equation between α and β; this equation will be that of the evolute.

◆

CONTACT AND INTERSECTION OF OSCULATING CURVES.

An osculating curve may be situated in two different ways in respect of the curve with which it is in contact: 1st. it may have its two branches, both of them above the curve, as BB, or both of them below, as CC, (fig. 14.) in which case the osculate will only touch the curve: 2d. the osculate may have one branch above and the other below the curve, as in fig. (13.) and in this case the osculate will cut the curve in the point P.

We will proceed to shew in what cases respectively these conditions take place.

For the same abscissa $x + h$

let Y be the ordinate of the curve,

Y_2 be the ordinate of the osculate,

we have then

$$Y=\phi\,(x+h)=\phi x+Ah+Bh^2+Ch^3+\&c.$$
$$Y_2=F\,(x+h)=Fx+A_2h+B_2h^2+C_2h^3+\&c.$$

Now if the osculate be of the second order, the three first terms of these developements will be the same; whence the difference of the ordinates, corresponding to $x+h$, will be

$$(C-C_2)h^3+,\ \&c. \ldots\ldots\ldots \quad (47).$$

Suppose now that the abscissa becomes $x-h$; we must then change h into $-h$ in the difference of the ordinates, which will become

$$-(C-C_2)\,h^3+,\ \&c. \ldots\ldots \quad (48)$$

and since by taking h small this term may be greater than all the succeeding terms, it follows that the difference of the ordinates will change its sign when the abscissa, instead of being $x+h$, becomes $x-h$.

Hence the osculate is on one side above the curve, and on the other below it, and consequently cuts it.

This conclusion applies to all osculates of an *even* order.

If the *osculate* be of an *odd order* it will *touch* the curve: for we shall then have $+(D-D_2)h^4$ in each series.

We have already seen that if we have $(n+1)$ equations

$$Fx=fx\frac{dFx}{dx}=\frac{d\phi x}{dx},\ \&c.$$

we shall have an osculation of the nth degree.

In *general* also it appears that the osculation of a circle with any curve is of the second degree; since the three constants of its equation give us *three* equations such as those above.

But in *particular cases* it may happen that we can obtain another such equation, and thus have the osculation at particular points of the third degree.

We have deduced the expression for the radius of curvature γ in terms of the first and second differential coefficients of the ordinate of the curve: which are respectively equal to those belonging to the circle of curvature.

Let the successive differential coefficients, to any number, in the curve be represented by

$$p \; q \; r \; s \; \&c.$$

and those in the circle by

$$h \; k \; l \; m \; \&c.$$

Then, since we have

$$\gamma = \phi \, (pq).$$

if we take its successive differential coefficients, we have

$$\frac{d\gamma}{dx} = \phi_2 \, (p \, q \, r)$$

$$\frac{d^2\gamma}{dx^2} = \phi_3 \, (p \, q \, r \, s)$$

$$\&c. \qquad \&c.$$

since we necessarily introduce a new term r, s, &c. at each successive differentiation.

In the circle, since γ is constant,

$$\frac{d\gamma}{dx} = 0 = \phi_2 \, (h \, k \, l)$$

$$\frac{d^2\gamma}{dx^2} = 0 = \phi_3 \, (h \, k \, l \, m)$$

$$\&c. \qquad \&c.$$

If besides the original supposition $p = h \;\; q = k$, we have also $r = l$; then in the curve,

$$\frac{d\gamma}{dx} = \phi_2 \, (h \, k \, l) = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (49)$$

which is the condition corresponding to $\gamma =$ a maximum or minimum according as $\frac{d^2\gamma}{dx^2}$ is $-$ or $+$.

But if we have also $s = m$, but not $t = n$,

$$\text{then } \frac{d^2\gamma}{dx^2} = \phi_3\,(h\,k\,l\,m) = 0 \;\ldots\ldots\; (50)$$

or the differential coefficient which vanishes being of an even order, γ belongs to a point which is neither a maximum nor a minimum.

Hence we also see conversely that if at any point in a curve the curvature be a maximum or a minimum, the osculation is of an odd order at that point.

Thus *in general* the circle of curvature has a contact of the *second degree:* but at the *particular* points in a curve, where the curvature is a maximum or minimum, it has a contact of *the third degree.*

APPLICATION OF THE DIFFERENTIAL CALCULUS TO POLAR CURVES.

In the application of the differential calculus to spirals, or to curves referred to polar equations in general, it is often desirable, instead of the equation which we obtain by the common formulæ for transformation, from the polar to the coordinate equation, to obtain one between x and y which shall *not contain the transcendental term,* $\tan.^{-1}\frac{x}{y}$, but which will *involve differentials;* and, for this purpose, representing by

$$F(\theta, r) = 0,$$

the equation which it is required to transform into a function of the rectangular coordinates x, y, it appears by the form for transformation, that the value of r may be expressed in terms of x and y, without any transcendental, but that the same cannot be done in respect to θ; on which account, therefore, we eliminate θ between the equation $F(\theta, r)=0$, and its differential, which we will represent by $F(\theta, r, d\theta, dr)=0$; this process will, in fact, introduce into the result the differentials $d\theta$ and dr; but we shall see that these differentials may be expressed in functions of the variables x, y, dx and dy.

For, first, the formulæ for transformation give us

$$\cos.\,\theta=\frac{x}{r},\ \sin.\,\theta=\frac{y}{r},$$

dividing the second of these equations by the first, we obtain

$$\frac{\sin.\,\theta}{\cos.\,\theta}=\frac{y}{x}\ \text{or}\ \tan.\,\theta=\frac{y}{x};$$

differentiating, there results

$$\frac{d\theta}{\cos.^2\theta}=\frac{xdy-ydx}{x^2};$$

and replacing $\frac{1}{\cos.^2\theta}$ by its value, derived from the first of the equations, and suppressing the common factor x, we find

$$r^2d\theta=xdy-ydx;$$

whence, consequently,

$$d\theta=\frac{xdy-ydx}{r^2};$$

and putting for r^2 its value, this equation becomes

$$d\theta=\frac{xdy-ydx}{x^2+y^2}.\ \ldots\ldots\ldots\ldots \quad (51)$$

The differential of the other variable is found still more easily, for

$$r=\sqrt{x^2+y^2};$$

which being differentiated, we have

$$dr=\frac{xdx+ydy}{\sqrt{x^2+y^2}}; \quad \ldots\ldots\ldots\ldots (52)$$

and, by means of these values of $d\theta$, dr, and r, we shall change the equation obtained by the elimination of θ into another involving only x, y, dy, and dx; and which, consequently, will be the equation belonging to the rectangular coordinates.

We have seen that the differential of an arc z, referred to rectangular coordinates, has for its expression,

$$dz=\sqrt{dx^2+dy^2}.$$

It may be proposed to find the differential of the same arc, when expressed by a polar equation; and, in this case, we must substitute in this equation the values of dx and dy, derived from the equations

$$x=r.\cos.\theta,\ y=r.\sin.\theta.$$

Now, by differentiating these equations, we shall find

$$\begin{aligned} dx &= -r\sin.\theta.d\theta+\cos.\theta dr, \\ dy &= r\cos.\theta\, d\theta+\sin.\theta dr; \end{aligned}$$

whence, squaring these last equations, and reducing them by means of the formula

$$\sin.^2\theta+\cos.^2\theta=1,$$

we shall obtain

$$dz=\sqrt{r^2d\theta^2+dr^2}; \quad \ldots\ldots (53)$$

which is the differential of the arc in a function of the polar variables.

ON SUBTANGENTS AND SUBNORMALS, TANGENTS AND NORMALS, TO POLAR CURVES.

In curves referred to rectangular coordinates, the subtangent MT (fig. 2.) is the line intercepted between the foot M of the ordinate, and the point T in which a perpendicular, MT to that ordinate meets the tangent. Retaining the same definition for polar curves, in which the ordinate is no longer PM, but the radius vector OP, the subtangent will be then the line OT, drawn *perpendicular* to OP, (fig. 17.) and comprised between the point O and the point T, in which that perpendicular cuts the tangent. The subtangent therefore has, in polar curves, a position different to that it has in curves referred to rectangular coordinates, since in these the subtangent is always measured along the axis of the abscissæ, whereas in polar curves, in which that axis no longer exists, the subtangent varies its position at every point of the curve.

To find the value of the polar subtangent. Taking two successive values of the radius, r and $r+k$ (fig. 16.) corresponding respectively to the angles θ and $\theta + \chi$, draw the secant joining their extremities: and from $r+k$ cut off r; and let the line joining this point with P be h; through the pole draw S parallel to h, and produce it to meet the secant. Then by similar triangles we have

$$\frac{s}{r+k} = \frac{h}{k};$$

also $h = r$. chord χ

$$\therefore \frac{s}{r(r+k)} = \frac{\text{chord. } \chi}{k}.$$

But k and chord χ vanish at the same time: hence when $k=0$ we have,

$$\frac{s}{r^2} = \frac{0}{0} = \frac{d\theta}{dr}.$$

Under this condition the secant becomes a tangent at P; s becomes the polar subtangent; and we have

$$\text{subtangent} = \frac{r^2 d\theta}{dr} \quad \ldots\ldots \quad (54).$$

To find the subnormal: since the normal is drawn perpendicular to the tangent, we have from the right angled triangles (m being the subnormal)

$$\frac{s}{r} = \frac{r}{m}, \quad \therefore m = \frac{r^2}{s};$$

or substituting the value of s,

$$m = \frac{r^2 dr}{r^2 d\theta},$$

or the subnormal $= \dfrac{dr}{d\theta}$ .... (55).

Also from the right angled triangles we have the values of n the normal and t the tangent,

$$n = \sqrt{r^2 + m^2}, \qquad t = \sqrt{r^2 + s^2};$$

and substituting in these equations the values of r, m, and s, we find

$$\text{normal} = \sqrt{\left(r^2 + \frac{dr^2}{d\theta^2}\right)} \quad \ldots\ldots\ldots\ldots \ (56)$$

$$\text{tangent} = r\sqrt{\left(1 + r^2\frac{d\theta^2}{dr^2}\right)} \quad \ldots\ldots\ldots \ (57).$$

ASYMPTOTES TO POLAR CURVES.

From the expression for the subtangent we obtain

$$\frac{s}{r} = \frac{rd\theta}{dr};$$

but from the right angled triangle, writing ϕ for the angle which the *tangent* forms with the *subtangent*,

$$\frac{s}{r} = \frac{s}{s \tan.\,\phi} = \frac{1}{\tan.\,\phi}$$

$$\therefore \frac{rd\theta}{dr} = \frac{1}{\tan.\,\phi}:$$

$$\text{or } \frac{dr}{d\theta} = r \tan.\,\phi \quad \ldots . \ (58).$$

From the value of which when $r = \infty$, we may derive the asymptote, as in coordinate curves.

To express the value of the perpendicular drawn from the pole upon the tangent.

We have obviously right angled and similar triangles formed by the radius, the normal, and the polar perpendicular, which last we may write $=p$; whence

$$\frac{p^2}{r^2} = \frac{r^2}{n^2}$$

$$=\frac{r^2}{\frac{r^2d\theta^2+dr^2}{d\theta^2}}=\frac{r^2d\theta^2}{r^2d\theta^2+dr^2}.$$

$$\text{Whence } p^2=\frac{r^4d\theta^2}{r^2d\theta^2+dr^2}$$

$$\therefore p=\frac{r^2d\theta}{\pm\surd(r^2d\theta^2+dr^2)}$$

$$=\frac{r^2}{\pm\surd\left(r^2+\left(\frac{dr}{d\theta}\right)^2\right)}\quad\ldots\ldots(59).$$

Some writers express polar curves by means of equations between r and p: and we shall find that in several cases this method is of great use.

DIRECTION OF CURVATURE IN POLAR CURVES.

If the equation to a spiral be expressed in terms of r and p, as in (59), it is evident when the spiral is *concave* (fig. 17.) towards its pole, as r increases or decreases, p increases or decreases, or the difference r_2-r has the same sign as p_2-p. If the spiral be *convex*, (fig. 18.) as r increases p decreases, or r_2-r has a different sign from p_2-p. Hence the ratio of these increments will in the first case be +, in the second −. But p being a function of r, we have the developement of $f(r+k)$ by Taylor's series; or,

$$f(r+k)=fr+\frac{dp}{dr}k+\frac{d^2p}{dr^2}\frac{k^2}{1.2}+\&c.$$

whence the ratio of the increments,

$$\frac{p_2 - p}{k} = \frac{dp}{dr} + \frac{d^2p}{dr^2}\frac{k}{1.2} + \&c.,$$

and since when k is small, the sign of the whole, that is, of the ratio of the increments, is the same as that of $\frac{dp}{dr}$, according to what was just observed,

$$\left.\begin{array}{l} \text{if we have } + \dfrac{dp}{dr} \text{ the spiral is concave,} \\ \ldots\ldots\ldots\ldots - \dfrac{dp}{dr} \ldots\ldots\ldots\ldots\ldots\ldots \text{convex.} \end{array}\right\} \ldots\ldots (60).$$

If the spiral change from concave to convex, $\frac{dp}{dr}$ must change its sign, or *at the point of inflexion* $\frac{dp}{dr} = 0$ or $= \infty$. Hence the values of r, which gives either of these conditions, will shew the point of inflexion.

ON THE DETERMINATION OF THE EXPRESSION FOR THE RADIUS OF CURVATURE IN POLAR CURVES.

The expression for the radius of curvature, referred to rectangular coordinates, assuming the positive sign for γ, is

$$\gamma = \frac{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

That this value of γ may be expressed in terms of the polar variables, we must eliminate the differential coefficients which enter into the formula by means of the following equations,

$$x = r \cos.\ \theta,\ y = r \sin.\ \theta;$$

which, being differentiated, and the results divided the one by the other, we shall obtain

$$\frac{dy}{dx} = \frac{dr \sin.\ \theta + r \cos.\ \theta\, d\theta}{dr \cos.\ \theta - r \sin.\ \theta\, d\theta};$$

and, representing the two terms of this fraction by m and n, we shall have

$$m = dr \sin.\ \theta + r \cos.\ \theta\, d\theta,$$
$$n = dr \cos.\ \theta - r \sin.\ \theta\, d\theta.$$

and consequently

$$\frac{dy}{dx} = \frac{m}{n}.$$

$$\frac{dy^2}{dx^2} = \frac{m^2}{n^2};$$

by means of which last equation we find for the numerator of the value of γ,

$$\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \left(\frac{m^2 + n^2}{n^2}\right)^{\frac{3}{2}};$$

and raising each term of this fraction to the power $\frac{3}{2}$, and observing that the power $\frac{3}{2}$ of n^2 is n^3, we have

$$\left(1 + \frac{d^2y}{dx^2}\right)^{\frac{3}{2}} = \frac{(m^2 + n^2)^{\frac{3}{2}}}{n^3}.$$

Differentiating now the equation $\frac{dy}{dx} = \frac{m}{n}$ we shall find

$$\frac{d^2y}{dx} = \frac{ndm - mdn}{n^2},$$

and dividing the first side of this equation by dx, and the second by n, which is equivalent to dx, we shall have

$$\frac{d^2y}{dx^2} = \frac{ndm - mdn}{n^3}.$$

By means of these values given by the two last equations, the expression for the radius of curvature becomes,

$$\gamma = \frac{(m^2 + n^2)^{\frac{3}{2}}}{ndm - mdn}.$$

and we have now only to transform this equation into a function of θ and r; for which purpose we must determine first the value of $n^2 + m^2$, by adding the squares of the value of m and n, and reducing by means of the equation $\sin.^2\theta + \cos.^2\theta = 1$; when we shall find

$$n^2 + m^2 = dr^2 + r^2\, d\theta^2.$$

To obtain the denominator of the equation for γ, we must differentiate successively the equations for m and n, considering $d\theta$ as constant; and multiplying the results by n and m respectively, we shall find

$$ndm = nd^2r \sin.\, \theta + 2ndr \cos.\, \theta d\theta - nr \sin.\, \theta d\theta^2,$$
$$mdn = md^2r \cos.\, \theta - 2mdr \sin.\, \theta d\theta - mr \cos.\, \theta d\theta^2,$$

whence, subtracting the second equation from the first, we have

$$\begin{aligned} ndm - mdn = d^2r\ &(n \sin.\, \theta - m \cos.\, \theta) \\ + 2drd\theta\ &(n \cos.\, \theta + m \sin.\, \theta) \\ -\ rd\theta^2\ &(n \sin.\, \theta - m \cos.\, \theta) \end{aligned}$$

Now, multiplying the first of the equations for m and n, by $\cos.\, \theta$, and the second by $\sin.\, \theta$, subtracting the one from the other, and reducing by means of the relation $\sin.^2\theta + \cos.^2\theta = 1$, we shall obtain

$$n \sin.\, \theta - m \cos.\, \theta = -rd\theta;$$

and the value of $n \cos. \theta + m \cos. \theta$ being determined in a similar manner, we shall find

$$n \cos. \theta + m \cos. \theta = dr.$$

Substituting these values in the equation above, it becomes

$$ndm - mdn = -rd^2rd\theta + 2dr^2d\theta + r^2d\theta^3,$$

and the values thus determined, change the equation for γ into

$$\gamma = \frac{(dr^2 + r^2d\theta^2)^{\frac{3}{2}}}{2dr^2d\theta - rd^2rd\theta + r^2d\theta^3} \ldots\ldots\ldots \quad (61)$$

PART II.

THE PRINCIPLES

OF THE

INTEGRAL CALCULUS.

THE

INTEGRAL CALCULUS.

General principles.

THE Integral Calculus is the inverse of the Differential. The object of it is, when a differential, or expression involving a differential is given, to find the quantity whose differential it is; this original quantity, considered as deduced from the differential, is called the *integral* of that differential. The process for finding it is called *integration;* and is usually represented by this notation:—

Suppose we have given

$$dy = dfx,$$

then we write,

$$y = \int dy = \int dfx.$$

the symbol $\int$ signifying the operation of obtaining the integral of the expression to which it is prefixed.

Expressions to be integrated may be classed under two general divisions.

I. Expressions which contain the differential of the variable involved *in such a form as arises from any of the processes of differentiation:* and which are therefore directly integrated by the inverse process.

Under this head may be included certain other expressions, which are easily reducible to this form; and these principally of two classes.

1st. Expressions reducible to the form of the logarithmic differential.

2d. Expressions reducible to the form of some of the differentials of trigonometrical functions.

All these expressions consisting of *one term*, we can refer to the same class expressions consisting of *a series of terms*, each of which separately belongs to this class; and consequently being separately integrated, the sum of the integrals gives the entire integral.

II. The second great division comprises expressions containing *a differential involved with some function of the variable, which does not arise from the process of differentiation*; and this class is subdivided according to the nature of such function.

The general expression will be

$$\int fx \, . \, dx,$$

where fx is either,

- I. Algebraic.
  - Rational
    - whole
    - fractional.
  - Irrational.
- II. Trigonometrical.
- III. Exponential.

Of Algebraic functions, those which are rational, in all cases admit of integration. For those which are irrational there are no general principles of integration: Many forms have never yet been integrated, and in the others much difficulty occurs. Certain classifications of them have however been effected, which enable us to integrate a considerable number of them.

INTEGRATION

OF

SIMPLE ALGEBRAIC DIFFERENTIALS.

The mode of obtaining the integrals of simple differential expressions, follows obviously from the principles of differentiation.

1st. Since $d\,(x \pm a) = dx$,
we have

$$\int dx = x \pm a,$$

which is usually written in a more general form

$$= x + C \quad \ldots\ldots\ldots\ldots\ldots \quad (1);$$

C representing a constant quantity, the value of which cannot be deduced from the given differential; but in particular instances may be determined from the conditions of the case.

This remark applies to all subsequent cases: where the addition of the constant, if not expressed, must be always understood.

2dly. Since $dax = adx$,

$$\int dax = a\int dx = ax \quad \ldots\ldots\ldots \quad (2).$$

3dly. Since $d\,(u\,.\,v\,\ldots\,z)$

$$= du\,.\,vz\,\ldots\, + dv\,.\,uz\,\ldots\ldots + dz\,.\,uv\,\ldots\,.$$

$$\therefore \int (du\,.\,vz + dv\,.\,uz, \&c.) = u\,.\,v\,.\,z\,.\,\&c. \ (3).$$

4thly, In like manner, since

$$d\frac{z}{y}=\frac{ydz-zdy}{y^2}$$

$$\therefore \int\left(\frac{ydz-zdy}{y^2}\right)=\frac{z}{y} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

5thly. Since $d\,(u \pm v \pm z, \&c.)$

$$= du \pm dv \pm dz, \&c.$$

$$\therefore \int(du \pm dv \pm dz, \&c.) = u \pm v \pm z, \&c. \ldots\ldots (5).$$

6thly. Since $dx^m = mx^{m-1}\,dx$, we have

$$\int mx^{m-1}\,dx = m\int x^{m-1}\,dx$$

$$\text{and} \therefore \int x^{m-1}\,dx = \frac{x^m}{m};$$

or, since the differential is the quantity which is first proposed to us,

$$\int x^m dx = \frac{x^{m+1}}{m+1} \quad \ldots\ldots\ldots\ldots\ldots (6).$$

This is usually enunciated in the following rule: to integrate $x^m dx$ *increase the index by unity, and divide by the index so increased, and by the differential.*

These are the most elementary cases, in which the integrals are immediately found; and several of the same rules may be extended to more complex cases. These we will now proceed to examine.

We may first remark, with respect to the constant, which is always to be added, that if the conditions of the case are such that the value of the integral corresponding to any particular value of the variable is known, then the value of the constant may be found.

Let the integral be $fx + C$, and suppose it known by the conditions of the question, that when $x = a$

$$fx = A \quad \therefore A = fa + C.$$

Whence $A - fa = C$

$\therefore$ the value of the integral is

$$fx - fa + A.$$

This is frequently simplified by knowing, that for the value $x = a$, the integral $= 0$.
And still further, if this value of x be $= 0$ also; in which case $C = 0$.

The consideration of the constant is then in all cases necessary for obtaining the complete value of the integral. If the integral be in the form $fx + C$, in which C is undetermined, it is called the *general or indefinite integral:* when in the form $fx - fa + A$, it is called the *particular or definite* integral.

If we suppose that the integral vanishes when $x = a$, the expression in this case becomes

$$0 = fa + C,$$

whence $C = -fa$.
And we have in general the integral $= fx - fa$.

The integral is in this case supposed to commence from the value 0 when $x = a$, and at any indefinite value of the variable x has the value assigned; viz. $fx - fa$.

If now we give to x a second determinate value b, then this *particular integral* will become

$$= fb - fa;$$

or we have the value of the portion of whole integral comprised between the values a and b of the variable: in this case we are said *to integrate between the limits* a *and* b.

There are one or two more complex cases which are readily reduced to the preceding rules.

1. Every polynomial, such as

$$(a+bx+cx^2+\ldots)^n\,dx,$$

where n is a positive integer, is integrated by merely expanding the polynomial, and then integrating each term separately.

2. Every expression, such as

$$(fx)^n.\,dfx$$

composed of two factors, of which one is the differential of the first power of the other, is integrated by writing

$$fx=z \text{ and} \therefore dfx=dz.$$

Whence by substitution

$$(fx)^n\,dfx=z^n dz,$$

and the integral is

$$\frac{z^{n+1}}{n+1}+C=\frac{(fx)^{n+1}}{n+1}+C \;\ldots\ldots\; (7).$$

3. The same principle will apply also if one factor differ from the differential of the other only as to the constants; as in the form

$$(a+bx^n)^p\,mx^{n-1}\,dx\,;$$

here we have

$$d\,(a+bx^n)=nbx^{n-1}\,dx:$$

$$\text{let } a+bx^n=z \quad \therefore dz=nbx^{n-1}\,dx,$$

$$\text{and } \frac{dz}{nb}=x^{n-1}dx.$$

Hence, by substitution, the original expression becomes

$$\frac{z^p m dz}{nb} = \frac{m}{nb} z^p dz;$$

the integral of which is

$$\frac{m}{nb} \cdot \frac{1}{p+1} \cdot z^{p+1} + C \ldots\ldots (8).$$

Which, substituting the value of z, gives the integral of the original expression.

INTEGRATION BY PARTS.

The elementary case of integrating the differential expression, which results from the rule for differentiating a product of two variables, gives rise to the method called *integration by parts*, which is of extensive use. If we have

$$d \,.\, uv = udv + vdu,$$

integrating and transposing we have

$$\int udv = uv - \int vdu \;\ldots\ldots\ldots\; (9).$$

This formula enables us to find the integral of *udv* from that of *vdu*, which in many cases we can obtain; and from which therefore we have the whole integral.

INTEGRATION OF THE LOGARITHMIC DIFFERENTIAL, AND OF EXPRESSIONS REDUCIBLE TO THIS FORM.

The rule for integrating $x^m dx$.
When $m = -1$ gives

$$\int x^{-1}\, dx = \frac{x^0}{0} + C = \infty,$$

but this form could not arise from differentiating $\frac{x}{0}$, which admits of no differential. We may however find the value by observing that $\frac{dx}{x}$ is the differential of log. x, or we have,

$$\int \frac{dx}{x} = \log.\ x + C. \quad \ldots\ldots\ldots (10).$$

It may here be proper to observe, that when the integral is a logarithm, the constant C may be introduced as a factor of the quantity whose logarithm we are considering.

For since the constant quantity must be equal to the logarithm of some quantity, we may write

$$C = \log.\ a\,;$$

hence we have the integral,

$$\begin{aligned} &= \log.\ x + \log.\ a \\ &= \log.\ (ax). \end{aligned}$$

To this form (10), certain other expressions are reducible, and which are therefore integrated by finding the logarithms.

1st. We have obviously

$$\int\frac{dx}{x+a}=\int\frac{d(x+a)}{x+a}$$

$$=\log.(x+a)+C. \quad \ldots\ldots\ldots\ldots (11).$$

2dly. In the same manner,

$$\int\frac{nx^{n-1}dx}{a^n+x^n}=\int\frac{d(a^n+x^n)}{a^n+x^n}$$

$$=\log.(a^n+x^n)+C \ldots\ldots\ldots\ldots (12).$$

3dly. The following differentials are also reducible to the same form;

$$\frac{adx}{a+bx}. \text{ Let } a+bx=z$$

$$\therefore dz=bdx \quad \therefore dx=\frac{dz}{b}$$

$$\frac{adz}{b(a+bx)}=\frac{adz}{bz}$$

$$\therefore\int\frac{adx}{a+bx}=\int\frac{a}{b}\cdot\frac{dz}{z}=\frac{a}{b}\log.z+C$$

$$=\frac{a}{b}\log.(a+bx)+C \quad \ldots. (13).$$

In the same manner,

$$\int\frac{adx}{a-bx}=-\frac{a}{b}\log.(a-bx)+C \quad \ldots (14).$$

In these cases the mode of proceeding is sufficiently obvious; in the following instances it is less so. They are usually called the logarithmic forms.

I. Let the given differential be

$$\frac{dx}{\sqrt{x^2\pm a^2}}.$$

$$\text{Let } x^2\pm a^2=v^2,$$

$\therefore$ taking the differentials $xdx = vdv$

$$\therefore \frac{dx}{v} = \frac{dv + dx = d(v+x)}{v+x}$$

$$\therefore \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log.(x+v) + C$$

$$= \log.(\sqrt{x^2 \pm a^2} + x) + C \;\ldots\ldots \quad (15).$$

II. Let the given differential be

$$\frac{dx}{\sqrt{x^2 \pm 2ax}}; \qquad \text{let } v^2 = x^2 \pm 2ax$$

$$\therefore vdv = (x \pm a)\,dx$$

$$\frac{dx}{v} = \frac{dx + dv = d(v + x \pm a)}{v + x \pm a}$$

$$\therefore \int \frac{dx}{\sqrt{x^2 \pm 2ax}} = \log.(x \pm a + \sqrt{x^2 \pm 2ax}) \;\ldots\ldots \quad (16).$$

III. Let the given differential be

$$\frac{2adx}{a^2 - x^2};$$

$$\text{this} = \frac{dx}{a+x} - \frac{-dx}{a-x}$$

$$\therefore \int \frac{2adx}{a^2 - x^2} = \log.(a+x) - \log.(a-x)$$

$$= \log.\frac{a+x}{a-x} \;\ldots\ldots\ldots\ldots \quad (17).$$

IV. If the form be $\dfrac{2adx}{x^2 - a^2}$,

we have, by a precisely similar process,

$$\int \frac{2adx}{x^2 - a^2} = \log.\frac{x-a}{x+a} \;\ldots\ldots\ldots\ldots \quad (18).$$

V. Let there be given $\dfrac{adx}{x\sqrt{a^2 - x^2}}$

and take $$a^2-x^2=y^2$$

$$\therefore x^2=a^2-y^2$$

$$\therefore \frac{axdx}{x^2\sqrt{a^2-x^2}}=a\cdot\frac{-ydy}{(a^2-y^2)y}=-\frac{ady}{a^2-y^2};$$

the integral of which by form III. is

$$-\tfrac{1}{2}\log.\frac{a+y}{a-y}=\tfrac{1}{2}\log.\frac{a-y}{a+y}$$

$$=\tfrac{1}{2}\log.\frac{(a-y)^2}{a^2-y^2}=\log.\frac{a-y}{x}$$

$$=\log.\frac{a-\sqrt{a^2-x^2}}{x}\ \ldots\ldots\ldots\ldots\ (19).$$

VI. If the form be $\dfrac{adx}{x.\sqrt{a^2+x^2}}$,

by a precisely similar process from form IV., we have the integral

$$=\log.\frac{\sqrt{a^2+x^2}-a}{x}\ \ldots\ldots\ldots\ldots\ (20).$$

THE INTEGRATION OF TRIGONOMETRICAL DIFFERENTIALS, AND OF EXPRESSIONS REDUCIBLE TO THIS FORM.

If z represent any circular arc, and we have

$$x=\sin.\ z,$$

then differentiating by (form 9), we obtain

$$dx=\cos.\ zdz,$$

whence we obtain

$$dz = \frac{dx}{\cos. z};$$

but the equation $\cos.^2 z + \sin.^2 z = 1$, gives us

$$\cos. z = \sqrt{1 - \sin.^2 z} = \sqrt{1 - x^2};$$

substituting this value, therefore, in that of dz, we obtain

$$dz = \frac{dx}{\sqrt{1 - x^2}},$$

and consequently we find, by integrating,

$$\int \frac{dx}{\sqrt{1 - x^2}} = z + C.$$

To determine the constant; since the arc z is 0 at the same time with the sine x, the equation will, on this hypothesis, be reduced to $0 = C$, and consequently

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin.^{-1} x. \quad \ldots\ldots\ldots\ldots \quad (21).$$

On the above integral may be made to depend that of

$$\frac{dx}{\sqrt{a^2 - x^2}};$$

for by dividing the two terms of the fraction by a, we shall have

$$\int \frac{\frac{dx}{a}}{\sqrt{1 - \frac{x^2}{a^2}}} = \int \frac{d.\frac{x}{a}}{\sqrt{1 - \frac{x^2}{a^2}}};$$

and since this integral is composed of $\frac{x}{a}$, in the same manner that $\int \frac{dx}{\sqrt{1 - x^2}}$, is of x, it follows that

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin.^{-1}\frac{x}{a}\ \ldots\ldots\ldots\ldots\ (22),$$

which is the form for the same arc to radius a.

Having thus the means of integrating the differential of a circular function, we may apply the same method whenever we have *an expression involving any variable* in the form

$$\frac{dx}{\sqrt{(a^2-x^2)}}.$$

The integration being effected by finding the value of the arc whose sine to radius $a=x$.

The same principle applies in all the subsequent cases of the trigonometrical functions: and expressions thus integrated are said to be *integrated by circular arcs.*

2dly. If we have $x=\cos.\ z$, by differentiating we obtain

$$dx=-\sin.\ z\ .\ dz.$$

$$\text{Whence, } dz=-\frac{dx}{\sin.\ z};$$

and putting for sin. z its value from the equation

$$\cos.^2x+\sin.^2x=1,$$

we shall obtain

$$dz=-\frac{dx}{\sqrt{1-\cos.^2 z}};$$

or, since $\cos.\ z=x$,

$$dz=-\frac{dx}{\sqrt{1-x^2}};$$

whence, integrating, we shall have

$$\int\frac{-dx}{\sqrt{1-x^2}}=\cos.^{-1}x+C.$$

To determine the constant, when $x=0$, $\int -\frac{dx}{\sqrt{1-x^2}}$ is also 0; in this case, then, the equation is reduced to

$$0=\cos.^{-1}0+C;$$

but in order that the cosine of the arc may be 0, that arc must become $=\frac{\pi}{2}$,

$$0=\tfrac{1}{2}\pi+C,$$

which gives us

$$C=-\tfrac{1}{2}\pi;$$

and substituting this value, we obtain

$$\int -\frac{dx}{\sqrt{1-x^2}}=-(\tfrac{1}{2}\pi-\cos.^{-1}x) \quad . \ . \ . \ . \quad (23).$$

3dly, If $x=\tan.\, z$, we find, by differentiating,

$$dx=\frac{dz}{\cos.^2 z};$$

and consequently

$$dz=dx\,.\cos.^2 z.$$

But since by Trigonometry,

$$\cos.\, z=\frac{1}{\sec.\, z},$$

we shall have

$$\cos.^2 z=\frac{1}{\sec.^2 z}=\frac{1}{1+\text{tang.}^2 z}=\frac{1}{1+x^2};$$

whence, substituting this value, we shall find

$$dz=dx\,.\frac{1}{1+x^2};$$

and, integrating, we shall have

$$\int\frac{dx}{1+x^2}=z+C.$$

And since when $x=0$, z becomes 0, and we have

$$0=C;$$

and therefore

$$\int\frac{dx}{1+x^2}=\tan.^{-1}x. \quad . \; . \; . \; . \quad (24).$$

We may bring under this form the integral

$$\int\frac{dx}{a^2+x^2};$$

for, dividing the two terms of the fraction by a^2, we may write it thus,

$$\int\frac{\frac{dx}{a^2}}{1+\frac{x^2}{a^2}}=\frac{1}{a}\int\frac{\frac{dx}{a}}{1+\frac{x^2}{a^2}};$$

and since this integral is composed of $\frac{x}{a}$, as $\frac{1}{a}\int\frac{dx}{1+x^2}$ is of x, we shall have

$$\int\frac{dx}{a^2+x^2}=-\frac{1}{a}\tan.^{-1}\frac{x}{a}. \quad . \; . \; . \; . \; . \quad (25)$$

which is the same form to radius a.

4thly. Let also x be the versine of z: then the cosine and versine being together equal to unity, we shall have

$$x+\cos.\, z=1;$$

and, by differentiating,

$$dx=dz\,.\,\sin.\, z;$$

whence we deduce

$$dz=\frac{dx}{\sin.\, z}:$$

but

$$\sin.\, z=\sqrt{1-\cos.^2 z}=\sqrt{(1-\cos.\, z)\,(1+\cos.\, z)}$$
$$=\sqrt{x(2-x)};$$

substituting, therefore, we have

$$dx = \frac{dx}{\sqrt{2x - x^2}};$$

and, by integrating,

$$\int \frac{dx}{\sqrt{2x - x^2}} = \text{versin.}^{-1} x. \quad . \; . \; . \; . \; . \quad (26)$$

We add no constant, because when x is 0, z is also 0.

INTEGRATION OF FORMS NOT ARISING FROM THE PROCESS OF DIFFERENTIATION.

We now proceed to the second great division of the Integral Calculus before pointed out: which is in fact by far the most extensive and important; but at the same time, in many parts involving investigations of a nature which will not fall within the plan of a work so completely elementary as the present. All that we shall attempt, will be to explain the fundamental principles of these processes; referring the reader, who is desirous of prosecuting the subject further, to the more extensive treatises already quoted.

The cases we have now to consider, are, as before stated, those of differentials affected by coefficients, which are some functions of the same variable, not arising from the process of differentiation.

The peculiar means of integrating such expressions will depend upon the different forms of the functions, and the means we have of reducing them to simple terms, which can be integrated by the methods already pointed out.

DIFFERENTIALS INVOLVED WITH ALGEBRAIC FUNCTIONS OF THE VARIABLE.

Of these we shall have two principal subdivisions; those which are rational, or involve the whole powers of the variable; and those which are irrational, or, in which the indices are fractional; and in each subdivision we shall have to distinguish the case where the coefficient is in the form of an integer, or of a fraction.

RATIONAL INTEGER COEFFICIENTS.

This case may be represented by the general form

$$(Ax^a + Bx^b + Cx^c + \ldots)\,dx \;.\;.\;.\;.\;.\; (27)$$

And any function which can be reduced to such a form with a finite number of terms is readily integrated, by integrating each term separately, and taking the sum.

RATIONAL FRACTIONAL COEFFICIENTS.

In the case in which the coefficient of the differential is a rational function of the variable in the form of a fraction, that is, containing the variable both in the numerator and denominator, we must have recourse to further considerations.

Since we shall, in what follows, have occasion to apply some propositions derived from the theory of equations, it may be convenient briefly to premise them.

It is easily shewn by the theory of equations; 1. that an equation which has only real and unequal roots is made up of factors of the form, $(x-a)$.

2. That if any number m of these roots be equal, the equation will have one factor of the form $(x-a)^m$.

3. If it have a pair of imaginary roots, it must contain a factor of the form $(x+\alpha)^2+\beta^2$.

4. If it have m, such pairs it must contain a factor of the form

$$\left((x+\alpha)^2+\beta^2\right)^m.$$

In order to investigate the means of integrating differential expressions involving a fractional coefficient containing whole powers of the variable, we will take, as the most general representation of such an expression, the form,

$$\frac{Px^m+Qx^{m-1}\;....+Rx+S}{P_2x^n+Q_2x^{n-1}\;....+R_2x+S_2}dx,$$

in which the multiplier of dx is a rational fraction. We will first shew that in the given expression, *we may always assume* n *greater than* m; for if this should not be the case the integration may be reduced to that of a differential of the same form, in which the highest power of x in the denominator is greater than the highest power of x in the numerator, by simply dividing the numerator by the denominator when arranged in *descending* powers of x, as in the following example:

Let the expression be

$$\frac{Px^3 + Qx^2 + Rx + S}{Q_2\, x^2 + R_2\, x + S_2},$$

dividing first by Q_2, we shall have

$$\frac{\frac{P}{Q_2}x^3 + \frac{Q}{Q_2}x^2 + \frac{R}{Q_2}x + \frac{S}{Q_2}}{x^2 + \frac{R_2}{Q_2}x + \frac{S_2}{Q_2}}$$

or, writing these fractional coefficients as single terms,

$$\frac{p\, x^3 + q\, x^2 + r\, x + s}{x^2 + r_2\, x + s_2}$$

But, performing the division, we should hence have a result in the form

$$px + m + \frac{kx + l}{x^2 + r_2 x + s_2},$$

and we should have to find the integral of this expression multiplied by dx, or,

$$\int (px + m)\, dx + \left(\frac{kx + l}{x^2 + r_2 x + s_2}\right) dx$$

which will be

$$\frac{px^2}{2} + mx + \int \left(\frac{kx + l}{x^2 + r_2 x + s_2}\right) dx$$

so that the integration is reduced to that of this last term, which is of the original form, having $n > m$.

Thus we may take as the most general form

$$\frac{Px^{m-1} + Qx^{m-2} + \ldots\ldots Rx + S}{x^m + Q_2 x^{m-1} + \ldots\ldots R_2 x + S_2} dx. \;\ldots\ldots\; (28).$$

The denominator will, by the theory of equations, be the product of m, factors $(x-a)$ $(x-b)$, &c.

* * *

Case I. *Let the roots be real and unequal.* Supposing we have solved the equation

$$x^m + Q_2x^{m-1} + \ldots + R_2x + S_2 = 0;$$

and having found that it is the product of the factors, $x-a$, $x-b$, $x-c$, &c. assume

$$\frac{Px^{m-1} + Qx^{m-2} \ldots + Rx + S}{x^m + Q_2x^{m-1} \ldots + R_2x + S_2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \&c.$$

Reducing then the second side of this equation to a common denominator, the numerator of each of these fractions will be multiplied by the product of the denominators of all the others, i.e. by a polynomial in terms of x of the order $(m-1)$; and the second side of the equation will consequently be a polynomial consisting of m terms. It follows, therefore, that if we equate the coefficients of the same powers of x on the two sides of the equation, we shall have m equations of condition for determining the m coefficients, A, B, C, &c.; and these coefficients being known, we shall then only have to integrate a series of terms such as

$$\frac{Adx}{x-a}, \frac{Bdx}{x-b} \&c.;$$

and the integral will therefore be

$$A \log. (x-a) + B \log. (x-b) + \&c. \ldots + C \ldots (29).$$

Case II. *Let the roots be real and equal.* In this case, therefore, the form will be

$$\frac{Px^{m-1}+\dots\dots}{(x-a)^m}dx \quad \dots\dots\dots\dots\dots\dots \quad (30).$$

If we took this in the same form as in the last case, we should have

$$\frac{A}{x-a}+\frac{B}{x-a}+\dots\dots$$

$$\text{or, } \frac{A+B+C+\dots\dots}{x-a}$$

where the numerater being in fact one constant, we should not have the (m) equations for determining the constants.

To avoid this we may proceed as follows:

Let us write the expression in this form,

$$\frac{A+Bx+Cx^2+\dots\dots+Px^{m-1}}{(x-a)^m}$$

This may be put under a form which will enable us to integrate. Let $x-a=z \quad \therefore x=a+z$, substituting this value, the expression becomes,

$$\frac{A+Ba+Bz+Ca^2+2\,Caz+Cz^2+\dots\dots}{z^m}$$

$$=\frac{A+Ba+Ca^2+\dots\dots}{z^m}+\frac{B+2\,Ca+\dots}{z^{m-1}}+\dots\dots$$

or, writing for the constant numerators, α, β, &c.

$$=\frac{\alpha}{(x-a)^m}+\frac{\beta}{(x-a)^{m-1}}+\&c.$$

Thus, from the constants in the given equation we have the values of α, β, &c. and we have only to integrate a series of terms,

$$\frac{\alpha\, dx}{(x-a)^m} \quad \frac{\beta\, dx}{(x-a)^{m-1}} \text{&c.}$$

which are all expressed by the general case

$$\int \frac{\alpha\, dx}{(x-a)^m} = \int \alpha\, dx\ (x-a)^{-m}$$

and since (replacing z) $dx = d\,(x-a) = dz$, this is integrated by the form (7.); which is,

$$\int fx^n.\, dfx = \frac{(fx)^{n+1}}{n+1} + C.$$

In this case then we have, by substituting in this form,

$$\int \alpha\, z^{-m} dz = \frac{\alpha\, z^{1-m}}{1-m} + C$$

$$= \frac{\alpha}{(1-m)\,(x-a)^{m-1}} + C \ldots (31).$$

Thus, if all the roots be equal, we have to integrate a series of terms all of this form. If only some are equal, the remaining unequal values may be integrated as in the former case; and the sum of the integrals taken.

◆

Case III. *Let the expression contain imaginary roots.*

1st. To begin with the simplest case, since imaginary roots enter equations by pairs, let us suppose the denominator of the fractional coefficient to contain *one pair of imaginary roots.* In this case, according to what was at first observed as arising from the theory of equations, we must have the expression of this form

$$\frac{Px^{m-1}+\ldots\ldots}{(x-a)(x-b)\ldots\ldots((x+\alpha)^2+\beta^2)}\cdot dx \ldots\ldots (32).$$

The last factor is the product of the two imaginary roots: let it be written $=(x-h)\,(x-k)$. Then, as in the first case, we have to write this fraction equal to the series of terms

$$\frac{A}{x-a}+\frac{B}{x-b}+\ldots\ldots+\frac{K}{x-h}+\frac{L}{x-k}.$$

But the last two terms, reduced to a common denominator, become

$$\frac{Kx-Kh+Lx-Lh}{(x-h)\,(x-k)};$$

which, collecting the terms, may be written

$$=\frac{Mx+N}{(x+a)^2+\beta^2}.$$

Thus we shall have the series of terms,

$$\frac{Adx}{x-a}+\frac{Bdx}{x-b}\ldots\ldots+\frac{Hdx}{x-h}+\frac{Mx+N}{(x+\alpha)^2+\beta^2}dx;$$

and having determined the constants A, B, ... H, M, N, by the process already employed, all these terms, except the last, will be integrated by logarithms; in respect to the last, it will be integrated in the manner following:
making $x+\alpha=z$, it becomes

$$\frac{Mz+N-M\alpha}{z^2+\beta^2}dz,$$

and representing the constant part $N-M\alpha$ by P, it is reduced to

$$\frac{Mz+P}{z^2+\beta^2}dz,$$

an expression which resolves itself into

$$\frac{Mzdz}{z^2+\beta^2}+\frac{Pdz}{z^2+\beta^2}.$$

To integrate the first, we must observe that zdz being the differential of $z^2+\beta^2$, with the exception of a constant factor, we may suppose $z^2+\beta^2=y$, which will give us, by differentiating,

$$zdz=\frac{dy}{2};$$

and substituting these values, we shall obtain $\frac{Mdy}{2y}$, whence the integral will be

$$\frac{M}{2}\log. y=\frac{M}{2}\log. (z^2+\beta^2)=\frac{M}{2}\log. [(x+\alpha)^2+\beta^2]$$

$$=\frac{M}{2}\log. (x^2+2\alpha x+\alpha^2+\beta^2)$$

$$=M\log. (x^2+2\alpha x+\alpha^2+\beta^2)^{\frac{1}{2}}.$$

In regard to the expression $\frac{Pdz}{z^2+\beta^2}$, by dividing the two terms by β^2, it may be brought under the form

$$\frac{P}{\beta}\cdot\frac{\frac{dz}{\beta}}{\frac{z^2}{\beta^2}+1},$$

and we see that its integral is

$$\frac{P}{\beta}\tan.^{-1}\frac{z}{\beta}=\frac{N-M\alpha}{\beta}\tan.^{-1}\frac{x+\alpha}{\beta};$$

lastly, therefore,

$$\int \frac{Mx+N}{x^2+2ax+\alpha^2+\beta^2}\,dx = M \log. \sqrt{x^2+2ax+\alpha^2+\beta^2}$$
$$+\frac{N-M\alpha}{\beta}\tan.^{-1}.\frac{x+\alpha}{\beta} \quad \ldots\ldots\ldots\ldots \quad (33).$$

2dly. Let us take the case where the denominator involves *several pairs of imaginary factors.*

When the fraction has in its denominator equal imaginary factors, it will contain one or more factors of the second degree, of the form $(x^2+2ax+\alpha^2+\beta^2)^p$, accordingly as it contains one or more groups of equal imaginary factors. The factor

$$(x^2+2ax+\alpha^2+\beta^2)^p$$

will correspond to the series of terms

$$\left.\begin{array}{l} \dfrac{H+Kx}{(x^2+2ax+\alpha^2+\beta^2)^p}+\dfrac{H_2+K_2x}{(x^2+2ax+\alpha^2+\beta^2)^{p-1}} \\ +\dfrac{H_3+K_3x}{(x^2+2ax+\alpha^2+\beta^2)^{p-2}}\ldots+\dfrac{H_k+K_kx}{x^2+2ax+\alpha^2+\beta^2} \end{array}\right\} (34),$$

and having proceeded in the same manner for the other groups of equal factors, we must determine the constants

$$H, K, H_2, K_2, H_3, K_3, \ldots H_k, K_k, \&c.$$

as before.

Multiplying then by dx, we shall have only to integrate each term separately, which may always be done if we know how to integrate the first term of the series multiplied by dx, since all the others are of the same form. For this purpose we shall put the term under the form

$$\frac{H+Kx}{[(x+\alpha)^2+\beta^2]^p}\,dx;$$

making $x+\alpha=z$, it will become

$$\frac{H-K\alpha+Kz}{(z^2+\beta^2)^p}\,dz\,;$$

and representing the constant part $H-K\alpha$ by M, we shall have to integrate

$$\frac{M+Kz}{(z^2+\beta^2)^p}\,dz,$$

a fraction which may be resolved into the two

$$\frac{Kzdz}{(z^2+\beta^2)^p}=\frac{Mdz}{(z^2+\beta^2)^p}.$$

To integrate the first, since zdz is the differential of $z^2+\beta^2$, except as to a constant factor, we will suppose $z^2+\beta^2=y$, whence we have $zdz=\frac{1}{2}dy$, and substituting, we shall obtain

$$\left.\begin{aligned}\int\frac{Kzdz}{(z^2+\beta^2)^p}&=\int\frac{1}{2}K.\frac{dy}{y^p}=\frac{1}{2}K\int y^{-p}dy\\&=\frac{1}{2}K.\frac{y^{-p+1}}{1-p}=\frac{1}{2}K\,\frac{(\beta^2+z^2)^{-p+1}}{1-p}\\&=\frac{1}{2}.\frac{K}{1-p}\,\frac{1}{(z^2+\beta^2)^{p-1}}+C.\end{aligned}\right\}\ldots(35)$$

It remains now to integrate $\dfrac{Mdz}{(z^2+\beta^2)^p}$, or $M(z^2+\beta^2)^{-p}dz$, to arrive at which integral, we shall deduce it from that of $\int(z^2+\beta^2)^p dz$, in the following manner:

We have obviously

$$(z^2+\beta^2)^p dz=(z^2+\beta^2)^{p-1}\,(z^2+\beta^2)\,dz\,;$$

or, performing the multiplication,

$$(z^2+\beta^2)^p dz=\beta^2\,(z^2+\beta^2)^{p-1}dz+(\beta^2+z^2)^{p-1}z^2dz,$$

and integrating, we shall have

$$\int(\beta^2+z^2)^p dz=\beta^2\int(z^2+\beta^2)^{p-1}dz+\int(\beta^2+z^2)^{p-1}z^2dz \quad (36).$$

Of the two integrals, on the second side of this equation, we shall leave the first under the integral sign; to the second we shall apply the method of parts. For this purpose, multiplying and dividing the expression $(\beta^2+z^2)^{p-1}z^2dz$ by 2, we shall put it under the form,

$$\frac{z}{2}(\beta^2+z^2)^{p-1}\,2zdz;$$

then $(\beta^2+z^2)^{p-1}\,2zdz$ will be the differential of

$$\frac{(\beta^2+z^2)^p}{p},$$

so that this expression will become

$$\frac{z}{2}d.\frac{(\beta^2+z^2)^p}{p},$$

and comparing it with the formula

$$\int udv=uv-\int vdu,$$

$$u=\frac{z}{2},\quad v=\frac{(\beta^2+z^2)^p}{p},$$

whence we shall find

$$\int\frac{z}{2}(\beta^2+z^2)^{p-1}2zdz=\frac{z}{2}\frac{(\beta^2+z^2)^p}{p}-\int\frac{(\beta^2+z^2)^p}{p}\frac{dz}{2}.$$

Substituting this value in place of the last term of the equation, and putting the constants without the integral sign, the equation will become

$$\int(\beta^2+z^2)^p dz=\beta^2\int(\beta^2+z^2)^{p-1}dz$$
$$+\frac{z}{2}\frac{(\beta^2+z^2)^p}{p}-\frac{1}{2p}\int(\beta^2+z^2)^p dz;$$

transposing the last term on the second side, and reducing, we shall find

$$\frac{1+2p}{2p}\int(\beta^2+z^2)^p dz = \frac{z}{2}\,\frac{(\beta^2+z^2)^p}{p} + \beta^2\int(\beta^2+z^2)^{p-1}dz\,;$$

which, dividing by $\frac{1+2p}{2p}$, becomes,

$$\int(\beta^2+z^2)^p dz = \frac{2p}{1+2p}\cdot\frac{z}{2}\cdot\frac{(\beta^2+z^2)^p}{p}$$
$$+\frac{2p\beta^2}{1+2p}\cdot\int(\beta^2+z^2)\,.^{p-1}dz.$$

Now recurring to the original form which we had to integrate, viz.

$$M\,(z^2+\beta^2)^{-p}.\,dz,$$

we shall be able to assign its value by the above expression if we change p into $-p$: by which it becomes

$$\int(\beta^2+z^2)^{-p}dz = \frac{z}{1-2p}(\beta^2+z^2)^{-p}$$
$$-\frac{2p\beta^2}{1-2p}\cdot\int(\beta^2+z^2)^{-p+1}.\,dz \;\ldots\ldots\ldots\ (37).$$

Here the quantity to be integrated in the last term is of the *same form* as the whole or quantity to be integrated on the first side, but having its index less by unity.

Hence for the integral of this last term we should have to substitute an expression of the same form as the whole, with the substitution of $-(p-1)$ for $-p$.

And here again the integral of the last term would be found in a similar way, the index becoming $-(p-2)$; thus the process would be continued till we had the last term with an index $= -1$. Thus the final expression to be integrated would be

$$(\beta^2+z^2)^{-1}dz=\frac{dz}{\beta^2+z^2}\ \ldots\ldots\ldots\ (38).$$

This is one of the forms which are integrated by circular arcs, and we have its integral

$$\frac{1}{\beta}\tan.^{-1}\frac{z}{\beta}.$$

* * *

From the preceding investigation we may remark, in general, that in each of the cases respectively, the process turns upon integrating expressions reducible to one or more of the following forms;

Case I. $\dfrac{Adx}{x-a}$

II. $\dfrac{Adx}{(x-a)^m}$

III. $\left\{\begin{array}{l}\dfrac{(Mx+N)\,dx}{x^2+a^2}\\ \dfrac{(Mx+N)\,dx}{(x^2+a^2)^p}\end{array}\right.$

And these integrations are all ultimately referred to one of the simple forms for powers, logarithms, or tangents.

Thus in all cases where the coefficient of a differential is a rational function of the variable, we can integrate the expression, supposing we can discover the factors from the multiplication of which will result the expression, which forms the denominator of the coefficient.

We have here investigated the modes of proceeding in their most general form: for further details the reader must refer to larger works.

INTEGRATION OF DIFFERENTIALS INVOLVED WITH THE VARIABLE IN IRRATIONAL FUNCTIONS.

Under this head there are often considerable difficulties in effecting the integration, and there are many cases which the skill of analysts has never yet succeeded in reducing to an integrable form. Nor have any general principles been discovered, though the principal cases which have been integrated have been classed under certain general forms, which we shall proceed briefly to explain.

We may however first observe, in general, that the principle of proceeding consists in this. The expression, which is an irrational function of x, may be a rational function of some function of x. Thus, suppose an irrational function, as $(fx)^{\frac{p}{q}}$, enter into the expression, then if we can find $(fx)^{\frac{p}{q}} = (\phi x)^n$ reducing the whole expression to terms of this second or rational function, we may integrate in terms of it, and then substitute its original value.

The principal forms of irrational functions which have been integrated, may be reduced to the following classes.

I. Those simple forms which we have already investigated by means of circular arcs.

II. Differential expressions in the form of a *rational function* of an assemblage of *functions of x*, some or all of which are themselves *irrational*, or denoting a rational function by the symbol F; this class may be represented by the form

$$F\left(x^a, x^{\frac{p}{q}}, x^{\frac{r}{s}} \ldots\right) dx \quad \ldots\ldots (39).$$

This expression is easily rationalized in the following manner:

Taking the fractional indices and reducing them to a common denominator, we have

$$\frac{sp}{sq} \qquad \frac{rq}{sq},$$

which we will write respectively equal to

$$\frac{v}{m} \qquad \frac{w}{m}.$$

And let $x = y^m \quad \therefore y = x^{\frac{1}{m}}$

$$\therefore y^v = x^{\frac{v}{m}} \quad y^w = x^{\frac{w}{m}}$$

$$\therefore dx = my^{m-1}\,dy.$$

Hence also the rational function

$$x^a = y^{ma}.$$

Then, by substituting these values, the whole expression becomes

$$F\,(y^{ma}, y^v, y^w)\; my^{m-1}\,dy \quad \ldots\ldots (40),$$

which is integrated as a rational function of y: and then the original value of y in terms of x substituted

III. Expressions of the form,

$$F\left(x, (A+Bx)^{\frac{p}{q}}, (A+Bx)^{\frac{r}{s}}\ldots\right) dx \ldots (41).$$

Here let $A+Bx=y$;

$$\text{whence } x=\frac{y-A}{B}$$

$$\therefore dx=\frac{dy}{B}.$$

Hence the expression becomes

$$\frac{1}{B}\cdot F\left(\frac{y-A}{B}, y^{\frac{p}{q}}, y^{\frac{r}{s}}\ldots\right).dy \ldots\ldots (42),$$

which reduces it to form II.

IV. Expressions of the form

$$F\left(x, \left(\frac{A+Bx}{\alpha+\beta x}\right)^{\frac{p}{q}}\ldots\ldots\right) dx \ldots\ldots (43).$$

Here, as in the last case, writing

$$\frac{A+Bx}{\alpha+\beta x}=y$$

we obtain

$$Bx-\beta xy=\alpha y-A,$$

$$\text{and } \therefore x=\frac{\alpha y-A}{B-\beta y},$$

by substituting which we may reduce this like the last case to form II.

V. Expressions of the form

$$F(x, \sqrt{A+Bx+Cx^2})\, dx \ldots\ldots (44).$$

Of which there may be two cases: the term Cx^2 may be positive or negative; if it be positive, we shall write the expression thus,

$$\sqrt{C}\sqrt{\frac{A}{C}+\frac{B}{C}x+x^2};$$

but if that term be negative, we shall consider it as the product of $+C$ by $-x^2$, and then the radical may be put under the form,

$$\sqrt{C}\sqrt{\frac{A}{C}+\frac{B}{C}x-x^2},$$

or putting, for greater simplicity,

$$\frac{A}{C}=a,\ \frac{B}{C}=b,$$

we shall have to integrate the two expressions

$$F(x,\sqrt{a+bx+x^2})\,dx,\ F(x,\sqrt{a+bx-x^2})\,dx.$$

1st. We will take the expression

$$F(x,\sqrt{a+bx+x^2})\,dx\ \ldots\ldots\ (45).$$

Here, in order to obtain, by a transformation, the values of x, dx, and $\sqrt{a+bx+x^2}$, in a rational function of a new variable z, we will suppose

$$\sqrt{a+bx+x^2}=z+x.$$

Raising this equation to the square, and suppressing the terms x^2 on each side, it becomes

$$a+bx=2xz+z^2,$$

whence we deduce

$$x=\frac{z^2-a}{b-2z}$$

and by means of this value we obtain

$$\sqrt{a+bx+x^2}=\frac{z^2-a}{b-2z}+z;$$

or, reducing to the same denominator,

$$\sqrt{a+bx+x^2} = -\frac{(z^2-bz+a)}{b-2z}.$$

It remains now to determine dx in terms of z, for which purpose differentiating the equation

$$a+bx=2xz+z^2$$

we have

$$bdx=2xdz+2zdx+2zdz,$$

whence we deduce

$$(b-2z)\,dx=2\,(x+z)\,dz,$$

and substituting the value before obtained,

$$x+z=-\frac{z^2-bz+a}{b-2z};$$

we have,

$$(b-2z)dx=-\frac{2(z^2-bz+a)}{b-2z}dz,$$

and therefore

$$dx=-\frac{2(z^2-bz+a)}{(b-2z)^2}dz.$$

Thus having obtained the values of

$$x,\ dx, \text{ and } \sqrt{a+bx+x^2},$$

by substitution the original expression becomes,

$$F\left\{\left(\frac{z^2-a}{b-2z}\right),\ \left(\frac{-(z^2-bz+a)}{b-2z}\right)\right\}$$
$$\left(\frac{-2(z^2-bz+a)}{(b-2z)^2}\right)dz \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;(46)$$

which, being a rational function of z, may be integrated by former methods.

2dly. Let us take the case

$$F\,(x,\ \sqrt{a+bx-x^2})dx \;.\;.\;.\;.\;.\;.\;.\;.\;(47).$$

Here the same mode of proceeding as in the former case will not apply; for if we should suppose $\sqrt{a+bx-x^2}=x+z$, on squaring the two sides of this equation, the terms x^2 would not destroy each other, but we should have a term $2x^2$, and the value of x in terms of z would result irrational. To treat this case, we must observe, that by the theory of equations the polynomial $a+bx-x^2$ may always be resolved into real factors of the first degree;

Let α and β be the roots of the equation

$$x^2-bx-a=0;$$

we shall have then, from the property of equations,

$$x^2-bx-a=(x-\alpha)(x-\beta),$$

and, consequently, by changing the signs,

$$a+bx-x^2=-(x-\alpha)(x-\beta)=(x-\alpha)(\beta-x);$$

substituting this value in the radical, we will suppose

$$\sqrt{(x-\alpha)(\beta-x)}=(x-\alpha)z;$$

which being squared, gives us

$$(x-\alpha)(\beta-x)=(x-\alpha)^2z^2,$$

and suppressing the common factor, we have

$$\beta-x=(x-\alpha)z^2,$$

whence we deduce

$$x=\frac{\beta+\alpha z^2}{z^2+1},$$

therefore,

$$x-\alpha=\frac{\beta+\alpha z^2}{z^2+1}-\alpha,$$

and reducing to the same denominator,

$$x-\alpha=\frac{\beta-\alpha}{z^2+1},$$

which value being substituted on the second side of the equation above, we obtain

$$\sqrt{(x-\alpha)(\beta-x)}=\frac{\beta-\alpha}{z^2+1}z.$$

In regard to dx, we have only to differentiate the equation to obtain its value in terms of z, and we shall have

$$dx=-\frac{2(\beta-\alpha)}{(z^2+1)^2}zdz.$$

Hence, making these several substitutions, the original form becomes

$$F\left\{\left(\frac{\beta+\alpha z^2}{z^2+1}\right), \left(\frac{\beta-\alpha}{z^2+1}z\right)\right\}\left(\frac{-2(\beta-\alpha)z}{(z^2+1)^2}\right)dz \ldots . (48),$$

which may be integrated as a rational function of z.

VI. Expressions of the form

$$F(x^{mn}, (A+Bx^n)^{\frac{p}{q}}, (A+Bx^n)^{\frac{r}{s}}, \ \ldots)x^{n-1}dx \ldots (49).$$

Such an expression is rationalized by reducing the fractional indices to a common denominator, which we will call k,

$$\text{and let } \frac{p}{q}=\frac{h}{k},$$

$$\text{and } A+Bx^n=z^k;$$

$$\therefore (A+Bx^n)^{\frac{p}{q}}=z^{k\frac{p}{q}}=z^h$$

$$\therefore x^n=\frac{z^k-A}{B}$$

and differentiating, we have

$$x^{n-1}dx=\frac{kz^{k-1}dz}{nB};$$

whence the expression becomes

$$F\left(\ \left(\frac{z^k-A}{B}\right)^m,\ z^h,\ z^l,\ .\ .\ .\ \right)\left(\frac{kz^{k-1}dz}{nB}\right)\ .\ .\ .\ .\ (50)$$

which is a rational function of z.

VII. Expressions of the form

$$F\left(x^{mn},\left(\frac{A+Bx^n}{\alpha+\beta x^n}\right)^{\frac{p}{q}},\ .\ .\ .\ .\ \right)x^{n-1}dx\ .\ .\ .\ .\ .\ .\ (51)$$

in which case the process is exactly the same as in the last.

VIII. Expressions of the form

$$x^m\,.\,F\left\{x^n,\ (A+Bx^n+Cx^{2n})^{\frac{1}{2}}\right\}dx\ .\ .\ .\ .\ .\ .\ .\ (52),$$

writing $x^n=z$,

we have $x=z^{\frac{1}{n}}$ $\quad x^m=z^{\frac{m}{n}}$,

and differentiating

$$dx=\frac{1}{n}\ z^{\frac{1}{n}-1}dz$$

and $x^m dx=\dfrac{1}{n}\ z^{\frac{m+1}{n}-1}dz$;

whence, by substitution, the expression becomes

$$\frac{1}{n}\,z^{\frac{m+1-n}{n}}\left(F\,(z,\sqrt{A+Bz+Cz^2}\right)dz\ .\ .\ .\ .\ .\ .\ (53),$$

which is the form V. and can therefore be integrated for z.

IX. The last form to be enumerated is

$$x^{m-1}(A+Bx^n)^{\frac{p}{q}}\ dx\ .\ .\ .\ .\ .\ .\ .\ .\ (54):$$

expressions of this form are called *binomial differentials*, and it is only in some cases that they can be rationalized.

But for an account of the investigation of them the reader must be referred to Dr. Lardner's Differential Calculus, p. 201, (from which the above enumeration of the forms of irrational functions is taken,) or to Boucharlat, p. 224.

INTEGRATION OF DIFFERENTIALS WHOSE CO-EFFICIENTS INVOLVE TRANSCENDENTAL FUNCTIONS OF THE VARIABLE.

The integration of expressions of this kind presents some difficulties, and has, in many cases, never yet been effected in finite terms.

The process depends upon the application of certain formulæ, peculiar to each sort of transcendental function, united with the methods already explained.

For the case of *exponential functions*, or,

$$\int a^x . fx . dx,$$

since we have

$$da^x = a^x . \log. adx,$$

we derive the form

$$\int a^x dx = \frac{a^x}{\log. a}.$$

This is the elementary integral of exponential functions; and it is by the combination of this with the integration of fx, that the complete integration is effected.

But to enter into this subject would be beyond the plan of this treatise. The reader must refer to Lardner, p. 228, or Boucharlat, p. 240.

Again; with respect to coefficients involving *trigonometrical functions*, a similar brief remark must here suffice.

For the integration of these functions there are *elementary integrals*, derived from the forms for the differentiation of each of them; by the combination of which with the other processes of integration many of these forms are integrated. See Lardner, p. 235: or Boucharlat, p. 235.

But *in all cases of transcendental functions*, we may obtain *an approximate integration* by series, according to a method about to be explained.

INTEGRATION BY SERIES.

After the foregoing imperfect sketch of the methods of integrating the various forms of differential expressions, it will here be the proper place to describe briefly the general methods of series.

We have already seen that when any expression can be resolved into a finite series of terms, each of which is a simple differential, we obtain the integral by taking the sum of the separate integrals of the terms. This principle is of very general application, as we will now proceed to explain.

In the expression Xdx let X be a function of x, which we can develope in the series,

$$Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + \ldots\ldots \quad (55).$$

Then we shall have $\int Xdx$,

$$\text{or,} \int\left(Ax^{\alpha}dx + Bx^{\beta}dx + Cy^{\gamma}dx + \ldots\right)$$

by taking the sum of the integrals of each term, or it =

$$\frac{Ax^{\alpha+1}}{\alpha+1} + \frac{Bx^{\beta+1}}{\beta+1} + \ldots\ldots \quad (56).$$

In this solution it is however to be observed, that should any of the indices α, β, &c. be $= -1$, the term so affected must be integrated by logarithms.

By this method, then, whenever the series terminates, we have the exact value of the integral. If it do not terminate we have still, *in all cases, an analytical expression for the integral*, but it will depend upon the circumstances of the case whether we can practically apply it: that is, it will depend upon the *convergency* of the series.

This will be the case when the powers are in an *ascending* series, if x represent a quantity less than unity.

If again the value of x be very great, and the powers descend or be negative, we shall also have an approximation to the value of the integral.

But in other cases we have no such approximation. Hence various methods have been sought and proposed for bringing such series into forms which will converge.

We will proceed to investigate one method which is very general, for obtaining *an approximate value of integrals by series.*

Let the expression to be integrated be Xdx, and suppose

$$\int Xdx = fx + C.$$

If by Taylor's theorem we develope the value assumed by fx when x becomes $(x+h)$, and take the difference or increment, the constant C being added to each expression, vanishes in the subtraction. And the value of the increment is the part of the integral included between the limits of the particular values of the variable x and $(x+h,)$ which being arbitrary, may be called a and b. If then we take the integral $fx+C$, and also develope its value when x becomes $x+h$ by Taylor's series, we have the difference

$$=\frac{dfx}{dx}\frac{k}{1}+\frac{d^2fx}{dx^2}\frac{h^2}{1.2}+\ldots\ldots$$

If now $x=a$ and $x+h=b\therefore h=b-a$, and the successive differential coefficients have determinate values which we will represent by A, B, &c.; thus we have the *value of the integral between the limits* a, b, represented by

$$A.\frac{(b-a)}{1}+B\frac{(b-a)^2}{1.2}+\ldots\ldots(57).$$

This series is the more convergent, the smaller we take the interval $b-a$.

If therefore a large interval $b-a$ be divided into any number of small parts, as n parts, each $=\alpha$, then for the integral between the limits a and $a+\alpha$, we have

$$\frac{A\alpha}{1}+\frac{B\alpha^2}{1.2}+\ldots;$$

between $a+\alpha$ and $a+2\alpha$

$$A_2\frac{\alpha}{1}+\frac{B_2\alpha^2}{1.2}+\ldots\ldots;$$

between $a+2\alpha$ and $a+3\alpha$

$$A_3\frac{a}{1}+\frac{B_3a^2}{1.2}+\ \ldots\ldots$$

Continuing to n such series, and taking their sum, they will compose the total value of the integral between a and b: or arranging by powers of a, we have the integral =

$$\left.\begin{array}{l} \frac{a}{1}\,(A+A_2+A_3\,\ldots+A_n.) \\ +\frac{a^2}{1.2}\,(B+B_2+B_3\,\ldots+B_n) \\ +\frac{a^3}{1.2.3}\,(C+C_2+C_3\,\ldots+C_n) \\ \quad\&c. \qquad\qquad \&c. \end{array}\right\}\ldots\ldots(58).$$

If we had taken a very small, we might neglect the powers above the first, or the integral would be expressed by

$$Aa+A_2a+A_3a+\ \ldots\ldots\ A_na\ \ldots\ (59).$$

For some further speculations connected with the subject of these series, the reader is referred to Lacroix, p. 274, &c.

This method is the invention of Euler.

BERNOUILLI'S SERIES.

In the method of integrating by series just explained, if we represent by Xdx a differential in which X is any function of x, we have *first to reduce the* *function represented by X into the form of a series*, and then to integrate.

In the method now to be investigated, we have the advantange of *reducing the expression* $\int X dx$ *into a series, without knowing the developement of* X.

In the form for integrating by parts, we have here to substitute $X = u$, $x = v$, and $dx = dv$, whence it becomes

$$\int X dx = Xx - \int x dX \quad \ldots\ldots (60).$$

In the last term we may write

$$dX = \frac{dX}{dx} dx;$$

and this last term may be integrated by the method of parts, by substituting

$$u = \frac{dX}{dx} \quad dv = xdx.$$

whence, since $xdx = \frac{dx^2}{2}$

we have $dv = \frac{dx^2}{2}$ and $v = \frac{x^2}{2}$.

Substituting these values, the formula gives

$$\int \frac{dX}{dx} xdx = \frac{dX}{dx} \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot d\left(\frac{dX}{dx}\right)$$

$$= \frac{dX}{dx} \cdot \frac{x^2}{2} - \frac{1}{2} \int x^2 \frac{d^2X}{dx}.$$

Here again in the last term we may write

$$\frac{d^2X}{dx} = \frac{d^2X}{dx^2} dx$$

and to integrate that term we have to substitute

$$\frac{d^2X}{dx^2}=u \quad x^2dx=dv$$

and $\therefore v=\frac{1}{3}x^3$.

Hence by the same formula

$$\int\frac{d^2X}{dx^2}x^2dx=\frac{d^2X}{dx^2}\cdot\frac{1}{3}x^3-\int x^3d\left(\frac{d^2X}{dx^2}\right)$$

$$=\frac{d^2X}{dx^2}\cdot\frac{x^3}{3}-\frac{1}{3}\int x^3\frac{d^3X}{dx^2},$$

the last term of which remains to be integrated in the same manner; and so on successively.

Thus making these successive substitutions in the original formula, it becomes

$$\int Xdx=Xx-\frac{dX}{dx}\frac{x^2}{1.2}-\frac{d^2X}{dx^2}\frac{x^3}{1.2.3}-\&\text{c.}+C\,..\,(61).$$

This series, which was the invention of John Bernouilli, corresponds in the Integral Calculus to Taylor's in the Differential.

ON SUCCESSIVE INTEGRATION.

From the process of successive differentiation it is evident, that if we have a differential coefficient of any order, as many successive integrations must be performed to arrive at the integral, or original function from which it was derived, as there are units in the exponent of its order; and the same number of arbitrary constants will be introduced.

From the nature of successive differentiation it is evident, that

$$\int \frac{d^n fx}{dx^n} = \frac{d^{n-1} fx}{dx^{n-1}} + A,$$

this second quantity being one, which differentiated and divided by dx, gives the first. Again,

$$\int\int \frac{d^n fx}{dx^n} = \int \frac{d^{n-1} fx}{dx^{n-1}} + A,$$

which, on the same principle,

$$= \frac{d^{n-2} fx}{dx^{n-2}} + Ax,$$

and again,

$$\int\int\int \frac{d^n fx}{dx^n} = \int \frac{d^{n-2} fx}{dx^{n-2}} + Ax$$

$$= \frac{d^{n-3} fx}{dx^{n-3}} + Bx + \frac{Ax^2}{2},$$

whence, in like manner,

$$\int\int\int\int \frac{d^n fx}{dx^n} = \frac{d^{n-4} fx}{dx^{n-4}} + Cx + \frac{Bx^2}{2} + \frac{Ax^3}{2.3}.$$

and so on successively; until we have

$$\int_n \frac{d^n fx}{dx^n} = fx + Nx + \frac{Mx^2}{2} + \frac{Lx^3}{2.3} + \ldots\ldots \quad (62).$$

In the brief and general account we have now given of the *integral calculus*, we have confined ourselves to the case of *functions of one variable*.

But there is a large and important class of expressions which require integration, involving *functions of two or more variables*.

The processes employed for this purpose are of a complex nature, and in their principle are analogous to the method before explained of *partial and total* differentials. Into this extensive branch of the subject, however, we shall not attempt to enter. The reader is referred to Lardner, p. 260, Boucharlat, p. 269, or Lacroix, p. 319.

Another important, extensive, and abstruse branch of the Calculus, is that which relates to *differential equations:* that is, equations involving two or more variables and their differential coefficients of any order. Such equations obviously take their rise from successively differentiating a common equation between two variables: but the solution of them, or the investigation of the integral equation from which they have been derived, is in general of considerable difficulty: and for the theory of it recourse must be had to the works already named.

APPLICATIONS

OF THE

INTEGRAL CALCULUS.

THE RECTIFICATION AND QUADRATURE OF CURVES.

RECTIFICATION OF CURVES.

To rectify a curve is to obtain a straight line equal in length to the arc of the curve. Now we found before, that the differential of the arc of a curve was expressed by

$$ds = \sqrt{dx^2 + dy^2}.$$

If, therefore, $y = fx$ be the equation of a given curve, we have only to find dy or dx, and substitute its value in the above formula; which will then contain only one variable under the radical sign: then if the expression be such as can be integrated, the curve is *rectifiable*, and we obtain

$$s = \int \sqrt{dx^2 + dfx^2} + C \quad \ldots\ldots (63).$$

for the length of its arc.

The rectification of an arc of a spiral is effected on precisely the same principles: for we have (Diff. Calc. form 53) in this case

$$dz = \surd(r^2 d\theta^2 + dr^2);$$

whence substituting for one of the variables, as a function of the other,

$$z = \int \surd(r^2 d\,(fr)^2 + dr^2) + C \, \ldots\ldots \, (64).$$

THE QUADRATURE OF CURVES.

In any curve referred to rectangular coordinates, (fig. 19.) the curvilinear area is a function of the abscissa and ordinate: and the ordinate being itself a function of the abscissa, the area is a function of the abscissa: and writing the values of the curvilinear areas corresponding to x, and to $x+h$, respectively, s and s_2, the latter may be developed as a function of $x+h$, or we have,

$$\text{the area } s_2 = s + \frac{ds}{dx}h + \frac{d^2s}{dx^2}\frac{h^2}{1.2} + \&\text{c.},$$

and we shall have therefore the increment

$$s_2 - s = \frac{ds}{dx}h + \frac{d^2s}{dx^2}\frac{h^2}{2} + \&\text{c.}$$

Also let the rectangle under the abscissa and ordinate, or

$$yx = xfx = p,$$

and that under $y(x+h)=(x+h)fx=p_2$, whence its increment, or $p_2-p=fx\,.\,h$.

Comparing these increments, we have

$$\frac{s_2-s}{p_2-p}=\frac{\frac{ds}{dx}h+\frac{d^2s}{dx^2}\frac{h^2}{2}+\&c.}{fx\,.\,h}=\frac{\frac{ds}{dx}+\frac{d^2s}{dx^2}\frac{h}{2}+\&c.}{fx}$$

but passing to the limit $s_2-s=$ (lim.) p_2-p. Hence when $h=0$, the above ratio becomes

$$\frac{0}{0}=1=\frac{ds}{dxfx},$$

whence $ds=fx.dx$; and putting for fx its value, we shall have

$$ds=ydx.$$

$$\text{or } s=\int ydx+C.\ \ldots\ldots(65).$$

We have then to deduce the value of dx from the equation of the curve, substitute it in the formula, and then integrate. We might equally substitute for y, fx, derived from the equation. We may in particular instances choose either method, according as the form resulting may be less complex.

We have here supposed that the coordinates are rectangular. If that be not the case, and the angle of ordination $=\omega$, we shall only have to modify the expression in the following manner.

The rectangular increment $p_2=p$ will now become an oblique parallelogram contained by the sides y and h. If, therefore, upon the side y a perpendicular be drawn from the extremity of h, this perpendicular $=h\,.\,\sin.\,\omega$. Hence we have,

$$p_2-p=y\,.\,h\,.\,\sin.\,\omega.$$

We have, therefore, only to substitute this value in the denominator of the expression, and the result arrived at as before will be,

$$s = \int y dx \sin. \omega + C. \quad \ldots . \; (66).$$

An integral such as $\int fx dx$ may always represent the area of a curve whose equation is $y = fx$. For this equation being given, if we substitute the value of y in the formula $\int y dx$, we shall have $\int fx dx$ for the area of that curve. It is on this account that when a problem conducts us to the integrating a function of only one variable, the problem is said to be reduced to quadratures.

According to what was observed before, the expression $\int y dx + C$ is a *general* or *indefinite* integral. But if we made the supposition that the integral should vanish when $x = a$, it would be a *particular* or *definite* integral.

To apply this to the quadrature of curves.

In making the hypothesis of the integral being 0 when $x = a$, we suppose that taking an abscissa $= a$, (fig. 20, 21.) the area is comprised between the ordinate at a, and the indefinite limit of the ordinate corresponding to x. A second limit will, in like manner, be fixed, if we give to x a determinate value b; and then the particular integral will give an expression for the area comprised between the ordinates at the given points a, b.

In this case we are said to effect the *quadrature between the limits* a *and* b.

To find the area of a sector of a spiral included between two radii and the curve.

(Fig. 22.) Describing a circular arc through any point P with radius r, cutting off the increment k from any other radius, forming with the first the incremental angle χ, we have the triangular increment,

$$\text{or } q = \frac{(r+k)\, r\,.\, \sin.\, \chi}{2}.$$

For the sector, or polar curvilinear area, we have,

$$s = f\theta,$$

$$s_2 = f(\theta + \chi),$$

$$= f\theta + \frac{ds}{d\theta}\chi + \frac{d^2s}{d\theta}\frac{\chi^2}{1.2} + \&c.$$

$$s_2 - s = \frac{ds}{d\theta}\chi + \&c.$$

Hence the ratio of the increments

$$\frac{s_2 - s}{p_2 - p} = \frac{\frac{ds}{d\theta}\chi + \&c.}{\frac{(r+k)\, r \sin.\, \chi}{2}} = \frac{\frac{ds}{d\theta} + \&c.}{\frac{(r+k)}{2}\,\frac{r \sin.\, \chi}{\chi}}.$$

But in the limit $\sin.\, \chi = \chi$, and the triangular increment = the curvilinear.

Thus when $h = 0$, this ratio becomes

$$\frac{0}{0} = 1 = \frac{\frac{ds}{d\theta}}{\frac{r^2}{2}} = \frac{2ds}{r^2 d\theta}$$

$$\therefore ds = \frac{r^2 d\theta}{2},$$

$$\text{or } s = \int \frac{r^2 d\theta}{2} \ldots\ldots (67).$$

DETERMINATION OF THE SURFACES AND CONTENTS OF SOLIDS OF REVOLUTION.

THE SURFACE OF A SOLID OF REVOLUTION.

If a curve, (fig. 19.) lying in one plane, revolve about the axis X, it will generate a solid of revolution. We will now investigate the expression for the differential of the surface of the solid which is thus generated.

For this purpose, taking

$$y = fx,$$

$$y_2 = f(x+h) = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.:$$

then the ordinates y and y_2 describing, in the course of their revolution, unequal circles, these circles will be the bases of a truncated cone, of which the chord z will be the side; and the expression for the surface of this truncated cone will be

$$\frac{\text{circ. } y + \text{circ. } y_2}{2} \times \text{chord. } z,$$

or, representing by π the ratio of the diameter to the circumference,

$$\frac{2\pi \,.\, y + 2\pi y_2}{2} \times \text{chord } z = \pi(y + y_2)\text{chord } z;$$

whence, putting for the ordinates y, y_2, their analytical values, we shall have surface of truncated cone z,

$$=\pi\left(2y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right)\text{chord } z,$$

and dividing by chord z,

$$\frac{\text{surface of cone } z}{\text{chord } z}=\pi\left(2y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right).$$

If now we represent by s the arc of the curve, and by u the surface generated by that arc; since, on diminishing h, the arc tends to coincide with the chord, the first side of the preceding equation must be replaced in the case of the limit, by the ratio of the increment of the curvilinear surface to that of the arc of the curve: hence, when $h=0$, we have

$$\frac{\text{incr}^t.\ u}{\text{incr}^t.\ s}=\frac{0}{0}=\frac{du}{ds}=2\pi y,$$

whence $du=2\pi y ds$; and putting for ds its value, we shall have, lastly,

$$du=2\pi y\sqrt{dx^2+dy^2},$$

$$\text{or } u=\int 2\pi y\sqrt{(dx^2+dy^2)}+C \ldots (68).$$

THE CONTENT, OR CUBATURE OF A SOLID OF REVOLUTION.

Let v be the volume of the solid generated by the revolution of the curvilinear area s, (fig. 19.), about the axis X.

If the abscissa x becomes $x+h$, this volume will be augmented by the part generated by the revolution of the trapezium s_2, included by the two ordinates, the increment h, and the portion of the curve intercepted, about the same axis; and since the volume generated by the revolution of the area s is a function of x, the volume generated by the revolution of $s+s_2$ will be a function of $x+h$, and will be expressed by

$$v_2=v+\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1.2}+\ldots\ldots$$

whence, subtracting the volume s, we shall have the solid increment s_2,

$$v_2-v=\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1.2}+\ldots\ldots$$

Now let $p=$ the cylinder generated by the revolution of the rectangle xy and p_2, that by $(x+h)y_2$: then the increment, or the cylinder y_2h will be

$$p_2-p=\pi(f(x+h))^2\,.\,h.$$

Comparing these ratios, we have

$$\frac{v_2-v}{p_2-p}=\frac{\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1.2}+\ldots\ldots}{\pi(f(x+h))^2\,.\,h}$$

$$=\frac{\frac{dv}{dx}+\frac{d^2v}{dx^2}\frac{h}{1.2}+\ldots\ldots}{\pi(f(x+h))^2},$$

but in the limit $v_2-v=(\text{lim.})p_2-p$:. hence when $h=0$, the ratio becomes,

$$\frac{0}{0}=1=\frac{\frac{dv}{dx}}{\pi(fx)^2}$$

whence we have

$$\frac{dv}{dx} = \pi(fx)^2 = \pi y^2,$$

$$\therefore dv = \pi y^2 dx,$$

$$\text{or } v = \int \pi y^2 dx + C. \ldots (69).$$

APPLICATION OF THE INTEGRAL CALCULUS TO THE DEVELOPEMENT OF CERTAIN FUNCTIONS.

I. Since we have

$$\log.(a+x)=\int\frac{dx}{a+x}=\int dx\,.\,\frac{1}{a+x}.$$

By developing we obtain

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&c.;$$

consequently,

$$\int\frac{dx}{a+x}=\int\left(\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&c.\right)dx;$$

whence, by integrating each term separately, we shall obtain

$$\int\frac{dx}{a+x}=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&c.+C,$$

Or we shall have

$$\log.(a+x)=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&c.+C.$$

To determine the constant, we must observe that when $x=0$, this equation is reduced to $\log. a=0+C$; which value being substituted for C, the equation will become

$$\log.(a+x)=\log.a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&c.$$

II. If we take the expression $\frac{dx}{1+x^2}$,

This differential being written thus, $\frac{1}{1+x^2} \times dx$, we have the developement

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \&c.;$$

and, therefore,

$$\int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c. \ldots + C,$$

which by circular arcs, is

$$\tan.^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c. \ldots + C.$$

When $x=0$, the arc becomes 0, and we have therefore $C=0$.

If the tangent be greater than unity, the terms of this series will go on increasing, and we cannot therefore give an approximate value of the arc; in this case, we shall obtain a converging series by putting $x = \frac{1}{x}$ in the equation, which will change it into

$$\frac{1}{1+\frac{1}{x^2}} = 1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6} + \&c.$$

whence, multiplying the two terms of the side by x^2, we shall have

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{x^2} + \frac{1}{x^4} - \frac{1}{x^6} + \&c.$$

and dividing by x^2, we shall obtain

$$\frac{1}{x^2+1} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8} + \&c.$$

consequently,

$$\int \frac{dx}{1+x^2} = \int \left(\frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8} + \&c. \right) dx\,;$$

and, performing the integrations separately,

$$\tan.^{-1} x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \&c. + C.$$

To determine the value of the constant, we must not put $x = 0$, for then the terms of the second side of equation will become infinite; but by making $x = \infty$, the expression $\tan.^{-1} x$ will become equal to the quadrant of the circle, and the equation becomes $\frac{\pi}{4} = 0 + C$; whence we have

$$\tan.^{-1} x = \frac{\pi}{4} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \&c.$$

III. Let there be given

$$\frac{dx}{\sqrt{1-x^2}} = dx\,(1-x^2)^{\frac{1}{2}}.$$

We can develope $(1-x^2)^{-\frac{1}{2}}$ by the binomial theorem in the following manner: calculating first the coefficients of the developements of $(1-x^2)^m$, on the hypothesis of $m = -\frac{1}{2}$, by writing down in order

$$m, \frac{m-1}{2}, \frac{m-2}{3}, \frac{m-3}{4}, \&c.,$$

and changing m into $-\frac{1}{2}$ in these expressions, when they will become

$$-\frac{1}{2},\ -\frac{3}{4},\ -\frac{5}{6},\ -\frac{7}{8},\ \text{\&c.}$$

and multiplying $-\frac{1}{2}$ by $-\frac{3}{4}$, that product by $-\frac{5}{6}$, and so on, we shall form the coefficients which are to be substituted in the place of A, B, C, &c. in the equation

$$(1-x^2)^{-\frac{1}{2}} = 1 - Ax^2 + Bx^4 - Cx^6 + \text{\&c.},$$

which will give

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}.\frac{3}{4}x^4 + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}x^6 + \text{\&c.},$$

and by integrating the equation

$$\frac{dx}{\sqrt{1-x^2}} = (1 + \frac{1}{2}x^2 + \frac{1}{2}.\frac{3}{4}x^4 + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}x^6 + \text{\&c.})\,dx,$$

we shall find

$$\sin.^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} = \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{x^7}{7} + \text{\&c.}$$

We put no constant, because when $x = 0$ the arc whose sign is x vanishes.

This formula will serve to determine an approximate value of the circumference; for by making $x = \frac{1}{2}$, it is reduced to

$$\sin.^{-1}\frac{1}{2} = \frac{1}{2} + \frac{1}{2}.\frac{1}{3}.\frac{1}{2^3} + \frac{1}{2}.\frac{3}{4}.\frac{1}{5}.\frac{1}{2^5} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{1}{7}.\frac{1}{2^7} + \text{\&c.}:$$

but the sine whose value is $\frac{1}{2}$, is equal to half the side of a regular hexagon, and answers to the twelfth part of the circumference; so that we shall have

$$\frac{\text{circumf.}}{12} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} \cdot \frac{1}{2^5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{7} \cdot \frac{1}{2^7} + \&\text{c.},$$

and consequently

$$\text{circumf.} = 12 \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} \cdot \frac{1}{2^5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{7} \cdot \frac{1}{2^7} + \&\text{c.} \right),$$

taking the ten first terms of the series

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2^3} + \&\text{c.},$$

we shall find for the sum

$$0{,}52359877,$$

and therefore

$$\frac{1}{2} \text{ circumference} = 6\,(0{,}52359877) = 3{,}14159262,$$

a value which is correct to the last figure of the decimals.

A
SUPPLEMENT
TO THE
ALGEBRAIC THEORY
OF
CURVES.

THE design of the ensuing Supplement is to render somewhat more complete the algebraic doctrine of Curves, particularly those of the second degree, as given in the former Treatise; by supplying some omissions, and furnishing the student with several additional investigations, easily deducible from the principles there laid down; which lead to more general views of the properties of the Conic Sections; and which will form a better preparative for the subsequent investigations of their other relations by means of the Differential and Integral Calculus.

◆

ADDENDA AND CORRIGENDA

IN THE

THEORY OF CURVES.

P. 23. l. 23. The two latter equations may be omitted; we only want

$$x_2 y = \phi \quad x x_2 = -\phi.$$

and in applying the formula for transformation, it must be recollected that we have here used the symbols x, y for the oblique, and x_2 y_2 for the rectangular coordinates. In the introduction they are used in the reverse sense. Hence the formula (Introd. 6.) which is

$$y = x_2 \sin.\, x x_2 + y_2 \sin.\, x y_2,$$

becomes

$$y_2 = -x \sin.\, \phi + y \sin.\, \phi.$$

P. 27. After line 3, add,

Since VV_2, now $= x_3 - x_2$, which, by (art. 6.), since $n = 0$, becomes

$$= \frac{1}{q} \sqrt{p^2} = \frac{p}{q}.$$

P. 28. l. 6. It will be understood that these two values are those of x_p, x_q, by taking half the sum of which, we obtain (in the next line) the value of x_m.

P. 33. l. 10. The expression would be better written

$$-a^2 + a^2 + 2ax + x^2,$$

and that in l. 12. should be

$$y^2 = \frac{p}{2a}(-a^2 + (a+x)^2),$$

and again, in l. 17.

$$y^2 = \frac{p}{2a}(-a^2 + x^2).$$

P. 37. art. 29. It may be remarked that the form

$$Bxy + Dy + Ex + F = 0,$$

when $D = 0$ and $E = 0$ becomes $Bxy = -F$,

$$\text{or, } xy = \frac{-F}{B}$$

in which, when $x = 0$ $y = \infty$, or the axes to which the locus is referred, are the asymptotes.

P. 42. l. 31. The three first steps of the solution may be more clearly expressed thus:

$$r^2 \,.\, \sin.^2\theta = \left\{ (1-e^2)\, a^2 - (1-e^2)\, a^2e^2 \right.$$
$$\left. -2(1-e^2)\, a\, e\, r \cos.\theta - (1-e^2)\, r^2 \cos.^2\theta, \right\}$$

or, $r^2 \sin.^2\theta + r^2\,(1-e^2)\cos.^2\theta + r\,.\,2\,(1-e^2)\, a\, e \cos.\theta$

$$= (1-e^2)(a^2 - a^2\, e^2)$$

$$r^2\,(\sin.^2\theta + \cos.^2\theta - e^2\cos.^2\theta) + r\,.\,2\,(1-e^2)\, a\, e \cos.\theta =$$

$$r^2\,(1 - e^2\cos.^2\theta) + r\,.\,2\,(1-e^2)\, a\, e \cos.\theta$$
$$= (1-e^2)\,(a^2 - a^2\, e^2)$$

$$r^2 + r\frac{2\,(1-e^2)\, a\, e \cos.\theta}{1 - e^2\cos.^2\theta} = \frac{(1-e^2)(a^2 - a^2\, e^2)}{1 - e^2\cos.^2\theta}.$$

P. 58. last line, read, "or dividing by r^2, *transpos-* "*ing* and extracting the root."

A SUPPLEMENT

TO THE

THEORY OF CURVES:

CONTAINING SEVERAL ADDITIONAL PROPERTIES AND INVESTIGATIONS.

FIRST DEGREE.

The following propositions are readily deduced from what is given in the body of the Treatise respecting lines of the first degree.

(1.) Since from the general equation we have

$$\frac{B}{A}=\frac{\sin. lx}{\sin. ly},$$

when the axes are rectangular, and the line passes through the origin, we have

$$\sin. lx=\cos. ly;$$

whence $\frac{B}{A}=\tan. lx$.

(2.) To express the equation to a straight line which shall pass through *two* given points.

Retaining the notation of art. 8, the two points are determined by the coordinates

$$x_2\, y_2 \qquad x_3\, y_3,$$

the equation of the line to pass through $x_3\, y_3$, is

$$A\,(y-y_3)+B\,(x-x_3)=0\,;$$

and since this line is also to pass through $x_2\, y_2$, the same equation must hold good when x and y have the particular values x_2 and y_2, or we have

$$A\,(y_2-y_3)+B\,(x_2-x_3)=0\,;$$

hence by division these equations become

$$\frac{A}{B}+\frac{x-x_3}{y-y_3}=0,$$
$$\frac{A}{B}+\frac{x_2-x_3}{y_2-y_3}=0\,;$$

whence by subtracting we have

$$\frac{x-x_3}{y-y_3}=\frac{x_2-x_3}{y_2-y_3},$$

$$\text{or } (x_2-x_3)\,(y-y_3)-(x-x_3)\,(y_2-y_3)=0,$$

which is the form required.

(3.) The expression (art. 7.) for tan. $l\lambda$ might be simplified, if we had assumed the equations to the two lines with the coefficients a, α, each $=1$; in which case the expression becomes

$$\tan.\, l\lambda=\frac{\sin.\,\omega\,(\beta-b)}{1+b\beta-(b+\beta)\cos.\,\omega}.$$

(4). To express the equation to a straight line which forms a given angle with a given straight line.

Let the equation of the given line be $y+bx+c=0$,

. . . . sought . . . $y+\beta x+\gamma=0$.

Let the tan. of the given angle $l\lambda=m$.

Then, from the above formula, by multiplication,

$$m + mb\,\beta - m\,(b+\beta)\cos.\,\omega = \sin.\,\omega\,(\beta - b)$$

$$m + mb\,\beta - mb\cos.\,\omega - m\beta\cos.\,\omega = \sin.\,\omega\,\beta - \sin.\,\omega\,b$$

$$\therefore (\sin.\,\omega + mb - m\cos.\,\omega)\,\beta = (\sin.\,\omega - m\cos.\,\omega)\,b - m$$

$$(\sin.\,\omega + m\,(b - \cos.\,\omega))\beta = \sin.\,\omega\,b - m\,(1 + b\cos.\,\omega)$$

$$\therefore \beta = \frac{b\sin.\,\omega - m\,(1 + b\cos.\,\omega)}{\sin.\,\omega + m\,(b - \cos.\,\omega)}.$$

which value substituted in the equation gives the line required, since $\gamma = 0$, by supposing it to pass through the origin, or may have any assumed value if it do not.

(5.) Cor. If the lines are to be at right angles

$$m = \infty,$$

and we have $\beta = \dfrac{-(1 + b\cos.\,\omega)}{b - \cos.\,\omega}$.

(6.) If the axes are also rectangular

$$\beta = -\frac{1}{b}.$$

SECOND DEGREE.

PROPERTIES OF THE TANGENT, SUBTANGENT, NORMAL, &c.

(1.) To find the equation to a tangent to a curve of the second degree at a given point in the curve.

The coordinates of the given point being x_2 y_2, the equation to *any straight line passing through that point* is (Curves I. 8.)

$$y-y_2=m\ (x-x_2).$$

The general equation of the second degree being

$$y^2=px+qx^2;$$

and this holding good for the point x_2 y_2, becomes

$$y_2^2=px_2+qx_2^2.$$

Whence by subtraction

$$y^2-y_2^2=p\ (x-x_2)+q\ (x^2-x_2^2),$$

$$\text{or, } (y-y_2)\ (y+y_2)=p\ (x-x_2)+q\ (x-x_2)\ (x+x_2).$$

And since at *any* point in which the straight line *meets* the curve the coordinates of the line and curve are identical, we may for any such point substitute the value of y from the equation to the straight line: or the above expression becomes

$$m\ (x-x_2)\ (y+y_2)=p\ (x-x_2)+q\ (x-x_2)\ (x+x_2);$$

by which we get rid of the factor $x-x_2$, and

$$m\ (y+y_2)=p+q\ (x+x_2).$$

$$\text{Whence } m=\frac{p}{y+y_2}+q\ \frac{x+x_2}{y+y_2}.$$

The equation to the straight line with this value of m belongs to a line meeting the curve in the point $x_2 y_2$, and also in any other point $x\, y$. If this point unite with the former, the two points of section merge in a point of contact, and the *equation* is that *of a tangent:* thus we have $x=x_2$ $y=y_2$

$$\therefore m=\frac{p}{2y_2}+q\frac{x_2}{y_2},$$

and the equation becomes

$$y-y_2=\left(\frac{p}{2y_2}+q\frac{x_2}{y_2}\right)(x-x_2).$$

This expression for m involving p and q, will of course assume a different form in each species of the curves. This consideration enables us to put the expression into forms convenient for finding the value of the subtangent, &c. in each curve.

1. In the parabola $q=0$, and the equation becomes

$$y-y_2=\frac{p}{2y_2}(x-x_2),$$

$$\text{or, } 2y_2\, y-2y_2^2=px-px_2;$$

substituting for $y_2^2=px_2$, this becomes

$$2y_2\, y=px+px_2.$$

At the point where the tangent meets the diameter we have in its equation $y=0$; hence in this case we have the corresponding value of $x=-x_2$

$$\therefore x+x_2=2x_2,$$

or the subtangent $=2$ the abscissa.

2. In the central curves, substituting the values

$$p=\frac{2b^2}{a} \text{ and } q=\frac{b^2}{a^2},$$

we have,

$$m=\frac{b^2}{a^2}\,\frac{x_2}{y_2}+\frac{b^2}{ay_2}=\frac{b^2}{a^2}\left(\frac{x_2}{y_2}+\frac{a}{y_2}\right).$$

Hence the equation becomes

$$y-y_2=\frac{b^2}{a^2}\left(\frac{a+x_2}{y_2}\right)(x-x_2):$$

whence

$$a^2y_2y-a^2y_2^2=b^2\,(a+x_2)\,x-b^2\,(a+x_2)\,x_2.$$

But from the equation, (§. ii. 24. c.), substituting the value of $a^2y_2^2=b^2\,(ax_2+x_2^2)+b^2ax_2$,
we have $a^2y_2y-b^2\,(a+x_2)\,x=b^2ax_2$.

At the point where the tangent meets the diameter, *in its equation,*

$$y=0\ \therefore x=\frac{ax_2}{a+x_2}.$$

Whence the *subtangent,* or $x+a$

$$=\frac{ax_2+(a+x_2)\,x_2}{a+x_2}=\frac{2ax_2+x_2^2}{a+x_2};$$

or the subtangent = the rectangle of the abscissæ divided by the abscissa from the centre. This expression evidently includes that for the parabola where $a=\infty$.

Hence is easily derived the following property. If t be the distance from the centre at which the tangent meets the diameter, we have

$$t=a\pm x\begin{cases}\text{Hyp.}\\ \text{Ell.}\end{cases}$$

or, substituting the value of x above,

$$t=\frac{a\,(a+x_2)\mp ax_2}{a\pm x_2}=\frac{a^2}{a\pm x_2};$$

or *the distance from the centre to the tangent* × *distance from centre to ordinate* = (*semi-diam.*)^2.

(2.) If we had supposed the origin at the centre, we should have had the general equation to the tangent for the central curves,

$$y - y_2 = \frac{b^2}{a^2}\frac{x^2}{y^2}(x - x_2).$$

Whence $a^2y_2y - a^2y_2^2 = b^2x_2x - b^2x_2^2$.
And from the equation to the curve

$$a^2y_2^2 = -b^2a^2 + b^2x_2^2;$$

whence the equation to the tangent becomes

$$a^2y_2y + b^2a^2 - b^2x_2^2 = b^2x_2x - b^2x_2^2;$$

$$\text{or, } a^2y_2y - b^2x_2x = -b^2a^2.$$

This form is sometimes used.

(3.) From the formula above given, in lines of the first degree, for the equation of a perpendicular to a given line (7), since we have the equation to a tangent to a curve of the second degree,

$$y - y_2 = m\ (x - x_2).$$

The equation to the *normal* is

$$y - y_2 = -\frac{1}{m}(x - x_2),$$

and when $y = 0$, we have

$$x - x_2 = my_2 = s;$$

which is the *value of the subnormal:* whence, by the right angled triangle, we have the *value of the normal:*

$$n = \sqrt{((x - x_2)^2 + y_2^2)} = \sqrt{(m^2y_2^2 + y_2^2)}.$$

In the parabola, $m=\frac{p}{2y_2}\ \therefore s=\frac{p}{2}$;

$$\text{also, } n=\sqrt{\left(\frac{p^2}{4}+y_2^2\right)}.$$

In the ellipse and hyperbola, taking the origin at the centre,

$$m=\frac{b^2}{a^2}\cdot\frac{x_2}{y_2}\quad\therefore s=\frac{b^2}{a^2}x_2$$

$$n=\sqrt{\left(\frac{b^4}{a^4}x_2^2+y_2^2\right)}.$$

(4.) In the ellipse and hyperbola we have directly from the focal construction, if r and r_2 be the lines from any point to the two foci, (as in the Conic Sections,)

$$r_2\pm r=2a\left.\vphantom{\int}\right\}\begin{array}{l}\text{Ell.}\\\text{Hyp.}\end{array}$$

(5.) If at that point y be an ordinate to the axis, ae, the excentricity, and the centre being the origin, we have in the right angled triangles

$$\begin{aligned}
r^2&=y^2+(ae-x)^2\\
r_2^2&=y^2+(ae+x)^2\\
\therefore r_2^2-r^2&=4aex\\
&=(r_2+r)(r_2-r)\\
&=2a\,(r_2\mp r)\\
\therefore (r_2\mp r)&=2ex
\end{aligned}$$

and since $(r_2\pm r)=2a$

subtracting, $\pm r=a-ex$

and adding, $r_2=a+ex$.

(6.) If s be the subnormal, we have the distance from the centre to the foot of the normal, or,

$$x \mp s = x \mp x\frac{b^2}{a^2} = x\left(1 \mp \frac{b^2}{a^2}\right)$$

$$= x\,(1 - 1 + e^2) = e^2 x.$$

Hence writing h and k for the distances from the two foci to the foot of the normal

$$h = ae - e^2x \qquad k = ae + e^2x$$

$$\therefore \frac{h}{k} = \frac{a - ex = \pm r}{a + ex = r_2}.$$

Thus the segments h k being as the sides r r_2, by Euc. VI. 3. the *normal bisects the angle* r r_2, or the adjacent exterior angle.

And consequently the radii r r_2 form equal angles with the tangent.

The following general properties of diameters may be easily obtained from the principles already laid down.

(7.) To find the equation to a diameter of an ellipse or hyperbola which passes through a given point.

Let the coordinates of the given point be x_2 y_2. Those of the centre (in the formula for a straight line through two given points) are $y_3 = 0$ $x_3 = 0$. Hence that formula gives as the equation to this diameter

$$x_2 y - x y_2 = 0.$$

Hence we may find the equation of the diameter g conjugate to this last, which we will call d.

From the Theory of Curves, (§. ii. 22.) we find, taking as the equation of the diameter, g,

$$\alpha y + \beta x = 0;$$

that of d will be $2A\beta y - 2C\alpha x = 0$;
but comparing this with its value deduced above,

$$2A\beta = x_2 \qquad 2C\alpha = y_2;$$

whence $\frac{\beta}{\alpha} = \frac{x_2 C}{y_2 A}$; but $\frac{C}{A} = q = \frac{b^2}{a^2}$,

$\therefore$ the equation to g becomes

$$a^2 y_2 y + b^2 x_2 x = 0.$$

(8.) To find the relation between the values of any two conjugate diameters, and those of the axes.

Let x_2 y_2 be the coordinates of the vertex of the semi-diameter d; then from the right angled triangle formed by the semi-diameter and the coordinates of its vertex, we have,

$$d^2 = x_2^2 + y_2^2,$$

or substituting the value of y_2^2 from the equation of the ellipse referred to the centre

$$d^2 = \frac{b^2}{a^2}(a^2 - x_2^2) + x_2^2$$

$$= b^2 - \frac{b^2}{a^2} x_2^2 + x_2^2$$

$$= b^2 + x_2^2\left(1 - \frac{b^2}{a^2}\right)$$

or since (Curves ii. 30.) $\frac{b^2}{a^2} = 1 - e^2$, by substituting this value the equation becomes

$$d^2 = b^2 + e^2\, x_2^{\,2}.$$

But the equation of the semidiameter g conjugate to d, being, as we have just found,

$$a^2y^2y + b^2x_2x = 0,$$

dividing each term by ab, we have

$$y\,.\,\frac{ay_2}{b} + x\,.\,\frac{bx_2}{a} = 0.$$

But this is (as above) the form of the equation to a diameter, the coordinates of whose vertex are

$$y_3 = \frac{-bx_2}{a} \qquad x_3 = \frac{ay_2}{b}.$$

Here again we have, therefore,

$$g^2 = x_3^{\,2} + y_3^{\,2} = \frac{a^2y_2^{\,2}}{b^2} + \frac{b^2x_2^{\,2}}{a^2};$$

which, substituting the value of $y_2^{\,2}$,

$$= a^2 - x_2^{\,2} + \frac{b^2}{a^2}x_2^{\,2} = a^2 + x_2^{\,2}\left(\frac{b^2}{a^2} - 1\right),$$

$$\text{or } g^2 = a^2 - e^2x_2^{\,2} : \ \therefore d^2 + g^2 = a^2 + b^2.$$

In the hyperbola we have in the same manner

$$d^2 = \frac{b^2}{a^2}\,(x_2^{\,2} - a^2) + x_2^{\,2},$$

which, as above, we find

$$= x_2^{\,2}\left(1 + \frac{b^2}{a^2}\right) - b^2;$$

and since here $\dfrac{b^2}{a^2} = e^2 - 1$, this becomes

$$d^2 = e^2x_2^{\,2} - b^2.$$

In like manner, for the conjugate diameter, (since the expressions for the coordinates of its vertex were not deduced, in the former case, on any assumption as to its being a point in the curve, and may therefore apply to the conjugate diameters of the hyperbola,) we have,

$$g^2 = x_2^2 - a^2 + \frac{b^2}{a^2} x_2^2$$

$$= -a^2 + x_2^2\left(1 + \frac{b^2}{a^2}\right)$$

$$= -a^2 + e^2 x_2^2.$$

Hence subtracting, we deduce,

$$d^2 - g^2 = a^2 - b^2:$$

or the *sum in the ellipse and the difference in the hyperbola, of any two conjugate diameters is equal to the sum or difference of the axes.*

(9.) Hence we may also deduce that *the sum* in the ellipse, *or the difference* in the hyperbola, of *the squares of the coordinates y* of the vertices of the conjugate diameters, is equal to the *square of the semi-conjugate axis:* and the sum or difference of the squares of the *coordinates x,* is equal to the square of the *semiaxis.*

For we have from the last article,

$$y_3^2 \pm y_2^2 = \frac{b^2 x_2^2 \pm a^2 y_2^2}{a^2},$$

and from the equation to the curves,

$$\pm a^2 y_2^2 = b^2 a^2 - b^2 x_2^2.$$

$$\therefore y_3^2 \pm y_2^2 = \frac{b^2 a^2}{a^2} = b^2.$$

Also we have $d^2 \pm g^2$, or

$$x_2^2 + y_2^2 \pm x_3^2 \pm y_3^2 = a^2 \pm b^2,$$

and adding or subtracting the last equation,

$$x_2^2 \pm x_3^2 = a^2.$$

(10.) Let the angle at which the diameter d meets the axis $= \phi$, and that at which g meets it $= \psi$.

Then from their respective equations (7) taking the coefficients of x in each, according to first degree, (1) we have

$$\tan. \phi = \frac{y_2}{x_2} \qquad \tan. \psi = \frac{b^2 x_2}{a^2 y_2},$$

whence $\tan. \phi \, . \tan. \psi = \frac{b^2}{a^2}$.

But this being $= q$ is $+$ in the hyperbola and $-$ in the ellipse.

Or the tangents of ϕ and ψ have the same sign in the hyperbola, but different in the ellipse; that is, the *angles lie in the same quadrant in the hyperbola, and in adjacent quadrants in the ellipse.*

If we have $\tan. \phi \, . \tan. \psi = 1$,

then $\tan. \phi = \frac{1}{\tan. \psi} = \cot. \psi$,

or $\phi = \frac{\pi}{4} - \psi$. Hence also $b = a$.

Thus the ellipse becomes a circle, and any conjugate diameters are at right angles.

The hyperbola becomes equilateral, and any two conjugate diameters form equal angles with the two axes and with the asymptote.

(11.) From the polar equation to the ellipse and hyperbola referred to the centre, (Curves, ii. 34.) we may deduce that *diameters which form equal angles with the axis, are equal:* since r in this case is any semi-diameter; and if in two values of r, cos. θ is the same, these values are equal.

The same polar equation also shews the maximum and minimum of the diameters. In the ellipse r is a maximum when the denominator $(1-e^2 \cos.^2 \theta)$ is a minimum; that is, when cos. θ is a maximum, or when $\theta=0$, r is a minimum when $\theta=\frac{\pi}{4}$.

In the hyperbola the equation becomes

$$r=a\sqrt{\left(\frac{e^2-1}{e^2 \cos.^2 \theta-1}\right)}$$

when $\cos.^2 \theta=\frac{1}{e^2}$, $r=\infty$, or becomes the asymptote.

if $\quad \cos.^2 \theta<\frac{1}{e^2}$, r is impossible.

if $\quad \cos.^2 \theta>\frac{1}{e^2}$, r is real.

when $\cos.^2 \theta=1$, or $\theta=0$, r is at its minimum.

From the first of these cases it follows, that since a negative value of r corresponding to the same angle θ on the other side of the axis would also be infinite, *the asymptotes form equal angles with the axis.*

Also, since in the case of the asymptote, we have

$$\cos.^2 \theta=\frac{1}{e^2},$$

and, consequently, $e^2=\frac{\cos.^2 \theta}{1}$, and, since $e^2-1=\frac{b^2}{a^2}$,

$$\therefore e^2-1=\frac{1-\cos.^2\theta}{\cos.^2\theta}=\frac{\sin.^2\theta}{\cos.^2\theta}=\tan.^2\theta,$$

or, $\tan.\ \theta=\frac{b}{a}$.

(12.) In the ellipse, if two *conjugate* diameters are equal, they form equal angles with the axis, by (11.) Hence $\phi=\psi$, and the expression, (10) becomes

$$\tan.^2\phi=-\frac{b^2}{a^2}$$

$$\therefore \tan.\phi=\frac{b}{a}$$

$$\tan.\ \psi=-\frac{b}{a}$$

or the equations of the equal conjugate diameters will be,

$$y+\frac{b}{a}x=0$$

$$y-\frac{b}{a}x=0$$

or they will coincide with diagonals of the rectangle described on the axes.

If an hyperbola have the same axes, we before found that the inclination of the *asymptote* was expressed by the same equation. Hence *the equal conjugate diameters of an ellipse coincide with the asymptotes of an hyperbola on the same axes.*

(13.) In general ϕ and ψ being the angles at which any conjugate diameters meet the axis, $(\phi+\psi)$ is the angle at which they are inclined to each other. Hence we have, by trigonometry,

$$\tan.(\phi+\psi)=\frac{\tan.\phi+\tan.\psi}{1-\tan.\phi.\tan.\psi}$$

In this expression since by (10) $\tan.\phi.\tan.\psi=$ a constant value, the denominator is constant, and the value of the expression is a minimum when the numerator is so; but in the ellipse $\tan.\phi$ and $\tan.\psi$ have different signs. Hence the numerator becomes $\tan.\phi-\tan.\psi$, which is evidently a minimum when $\tan.\phi=\tan.\psi$; Or *that pair of conjugate diameters which* form equal angles with the axis, and which consequently by (11) *are equal, form the least angle with each other.*

In the hyperbola it is manifest from what was before shewn, that the case $\tan.\phi=\tan.\psi$ is that of the conjugate diameters coinciding with each other, and consequently with the asymptote.

(14.) In the Parabola we before found the equation to a tangent at the point $x_2\, y_2$, to be

$$y-y_2=\frac{p}{2y_2}(x-x_2)$$

and ϕ being the angle at which the tangent meets the axis, (and which is also the angle at which the diameter through $x_2\, y_2$ meets its ordinates,)

$$\tan.\phi=\frac{p}{2y_2}$$

which may be written

$$\frac{\sin.\phi}{\cos.\phi}=\frac{p}{2y_2}$$

whence, $2y_2 \sin. \phi = p \cos \phi$,

and squaring, $4y_2^2 \sin.^2 \phi = p^2 \cos.^2 \phi$,

Substituting, $4px_2 \sin.^2 \phi = p^2 (1 - \sin.^2 \phi)$.

$$\left.\begin{array}{l}\text{Dividing by } p^2\\ \text{and transposing}\end{array}\right\} \left(\frac{4x_2}{p} + 1\right)\sin.^2 \phi = 1$$

$$\therefore \sin.^2 \phi = \frac{p}{4x_2 + p}.$$

If r_2 be the focal distance of the point x_2 y_2 and p_2, the parameter belonging to that point, we have

$$p_2 = 4r_2,$$

And from the construction of the curve by the directrix,

$$r_2 = x_2 + \frac{p}{4},$$

whence, $p_2 = 4x_2 + p$,

and substituting this value in the former equation, we have

$$\sin.^2 \phi = \frac{p}{p_2}$$

With any other point, when the parameter is p_3, we should have in like manner,

$$\sin.^2 \phi_2 = \frac{p}{p_3}$$

whence, $$\frac{\sin.^2 \phi}{\sin.^2 \phi_2} = \frac{p_3}{p_2}$$

or in the parabola the *parameters* of any diameters are *inversely as the squares of the sines of their angles of ordination.*

(15.) The following property is only a particular case of that which is shewn generally in curves of the nth degree, Sect. V. 11. but on account of its peculiar importance in connection with the properties of the curves of the second degree as deduced from the cone, it may be proper here to give it. It will be seen to include all the general properties investigated in Dr. Robertson's Conic Sections, as well as those cases given in the first part of the Elements of Curves.

The general equation of the second degree, if $y=0$, is reduced to

$$Cx^2+Ex+F=0.$$

This corresponds to the intersection of the locus with the axis Y.

Hence $$x^2+\frac{E}{C}x=-\frac{F}{C},$$

which we may write $x^2+2cx=d$,

and solving the equation, $x=-c\pm\sqrt{d+c^2}$.

The two values of x belong to two points of intersection, and these values multiplied together give

$$\begin{array}{c} -c+\sqrt{d+c^2} \\ -c-\sqrt{d+c^2} \\ \hline c^2-d+c^2 \end{array}$$

or, writing the two values a, a_2,

$$a\,.\,a_2=-d=\frac{F}{C}.$$

In like manner, if in the general equation x is $=0$, we have two values of y corresponding to intersections with the axis X, and by a similar notation,

$$b\,b_2=-d_2=\frac{F}{A},$$

Hence the ratio of these products, or

$$\frac{a\,a_2}{b\,b_2}=\frac{FA}{CF}=\frac{A}{C},$$

or is a constant ratio: being that of the coefficients of x^2, y^2 in the original equation.

In the form for transformation it is evident, that in the most general case we have to substitute for x and y expressions involving x_2 and y_2 with new coefficients: that is, the values of the coefficients of x_2^2 and y_2^2 in the equation will be different from those in the former.

When, however, the new axes are *parallel* to the former, we have only to substitute x_2+x_0 and y_2+y_0, in which case the coefficients of x_2^2 and y_2^2 will remain unaltered.

If therefore we now suppose the same locus referred to axes *parallel* to the former, and still intersecting them in points corresponding to two values

$$\alpha\,\alpha_2 \quad \beta\,\beta_2,$$

we shall by precisely the same process have the ratio

$$\frac{\alpha\,\alpha_2}{\beta\,\beta_2}=\frac{A}{C},$$

or in the same constant ratio as before.

Since the axes in each case may be considered as any straight lines intersecting the curve, and meeting each other *parallel* respectively to any other two secants, we have the general property that the *rectangles under the segments of the two first are always proportionals to the rectangles under the segments of the two last secants.*

(16.) The properties here given of lines of the second degree, combined with those investigated in the body of the tract on the theory of curves, will be sufficient to enable the student to form a complete system of the properties of the Conic Sections grounded upon these algebraical principles. To assist in making such deductions we will subjoin a brief synopsis of the several fundamental principles here established, and the order in which the differential properties follow from them.

1. The equation of the curves as referred to any diameter. Hence follow

The properties of rectangular diameters, and the formation of the curves in the cone.

Two curves of the same kind, on the same axis, with the same vertex, have their ordinates through the same point in a given ratio.

Hence the construction of the ellipse by the trammel.

2. The value of the subtangent.

Hence the semi-diameter a mean proportional to the segments.

Whence in the hyperbola the properties of conjugate ordinates, [C. S. I. i. 39]; but these are of no importance. Hence also (C. S. I. i. 44.) the (semi-diam.)^2 = rectangle of the segments of the parallel, or vertical tangents. The use of this property will, however, be superseded.

3. The focal construction of the curves and their polar equations.

Hence, sum or difference of focal lines = axis. And focal lines form equal angles with the tangent.

These properties may be deduced either geometrically, or by means of the normal, (as in this supplement).

In the parabola.

The bisection of the angle follows, either as in the Conic Sections, or from the subtangent.

The normal and subnormal, &c. either geometrically, or as in the supplement. Whence all the remaining properties follow directly. C. S. I. ii. (1—11.)

In the central curves the properties of the focal perpendiculars, normal, &c. follow directly. (21—27).

The subsequent articles may be deduced either as in the Conic Sections, by the help of the property [i. 44.], or by this supplement; having established the property,

$$a^2 \pm b^2 = d^2 \pm g^2 \ (34),$$

we deduce

$$d^2 = r \,.\, r_2 \ [29],$$

and thence [30—33].

Art. (38.) follows from the equation of the curve and (29).

Art. (39), or the ratio e = determining ratio, is included in the polar equation: from whence also in all the curves follows the property, that the *semi-latus rectum is an harmonic mean between the segments of any chord through the focus.*

This is given in C. S. Part II. ii.

It may perhaps be an assistance to the student to have this deduction given algebraically, and in a form somewhat more perspicuous.

1. The quantities a, b, c, are said to be in harmonical progression, if we have

$$\frac{a}{c} = \frac{a-b}{b-c}.$$

Hence $$ab-ac=ac-bc;$$
and dividing by abc

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}$$

$$\therefore \frac{1}{c}+\frac{1}{a}=\frac{1}{b}+\frac{1}{b}=\frac{2}{b}.$$

In the polar equation of the second degree, if r r_2 be two values of r, *lying in the same straight line* on opposite sides of the focus, they correspond to the geometrical description of the segments of any chord through the focus; and these forming angles in alternate quadrants, the signs of the cosines will be different: thus the sum of their reciprocals will be

$$\frac{1}{r}+\frac{1}{r_2}=\frac{1+e\cos.\theta+1-e\cos.\theta}{\frac{1}{2}p}=\frac{2}{\frac{1}{2}p}.$$

Consequently, by what was premised respecting harmonic progression, the quantities r, $\frac{1}{2}p$, and r_2, are in such a progression.

We have also made use of the polar equation referring to the centre to trace some general properties of the diameters, &c. which are not investigated in the Conic Sections.

FOURTH DEGREE.

Let it be required to trace the curve whose equation is

$$y^4 + 2ay^2x - ax^3 = 0.$$

Whence, $$y^4 + 2ay^2x = ax^3,$$

and solving this as a quadratic, we have

$$y^2 + ax = \pm\sqrt{(ax^3 + a^2x^2)}$$

$$\therefore y = \pm\sqrt{[-ax \pm \sqrt{(ax^3 + a^2x^2)}]}.$$

Here a being supposed + with $+x$, two values of y are impossible.

When $x = 0$ $y = 0$, and four values of y vanish.

If $-x = a$ $\quad y = \sqrt{(a^2 \pm \sqrt{(-a^4 + a^4)})}$

$= \pm a$, and the four values merge in two.

If $-x > a \ldots -ax^3 > +a^2x^2$ $\therefore y$ is impossible.

$-x < a \ldots\ldots xx^2 < ax^2$, or $x^3 < ax^2$

$\therefore -ax^3 < a^2x^2$,

or all the four values of y are real.

Hence the locus cuts the axis at the origin, so that four values of y vanish on the negative side, and two on the positive. (Diff. Calc. Fig. 12.) On the positive side it extends *ad infinitum* in two branches.

On the negative side between O and a the locus cuts the ordinate y four times, and the parallel through a forms a tangent at two points, one above, the other below X; at which points $y = \pm a$. The curve $\therefore$ forms two nodes, and has a triple point at the origin.

*N*TH DEGREE.

PARABOLÆ AND HYPERBOLÆ OF ALL DEGREES.

The statement respecting the infinite branches of the *n*th parabolæ and hyperbolæ, (Sect. V. 8.) may require this preliminary observation.

If in the general form $y^n = px^m$ m and n are *both even numbers*, it is evident that they are both divisible by 2, or some multiple of 2, until *at least one* of them result an *odd* number: and the indices must be thus reduced, in order that the equation may be in its lowest terms, so as to give it in its real dimensions; the index of the constant being immaterial.

It is therefore supposed that the equation is thus reduced before we apply the considerations in Art. 8. Thus the several cases (which apply either to $+$ or $-m$) may be more generally and perspicuously stated, as follows:

$$\begin{array}{ll}
m \text{ even} & n \text{ even} \\
\pm x \text{ gives} + px^m \ldots \therefore & +y^n \therefore +y \\
\begin{array}{c} m \text{ odd} \\ \pm x \ldots\ldots \pm px^m \ldots \therefore \end{array} & \left\{ \begin{array}{l} n \text{ odd} \\ \pm y^n \therefore \pm y \\ n \text{ even} \\ \left\{ \begin{array}{l} +y^n \therefore \pm y \\ -y^n \text{ imposs.} \end{array} \right. \end{array} \right.
\end{array}$$

The following remarks on these curves may also be added.

1. In the general equation

$$y^n = px^{\pm m}.$$

If we have $+m=n$, by extracting the root on both sides we deduce

$$y = p^{\frac{1}{m}}x,$$

or the parabola becomes a straight line.

If we have $-m=n$, in the same way

$$yx = p^{\frac{1}{m}},$$

or the equation is that of the common hyperbola.

2. In general we have for the hyperbolæ,

$$y^n = \frac{p}{x^m} \text{ and } \therefore y = \frac{p^{\frac{1}{n}}}{x^{\frac{m}{n}}}$$

$$x^m = \frac{p}{y^n} \text{ and } \therefore x = \frac{p^{\frac{1}{m}}}{y^{\frac{n}{m}}}.$$

If $m > n$ and therefore (*a fortiori*) $\frac{m}{n} > \frac{n}{m}$, $x^{\frac{m}{n}}$ increases faster than $y^{\frac{n}{m}}$.

Hence from the above values of y we deduce, that when $m > n$, y varies more rapidly than x, or *the curve approaches more rapidly towards the asymptote X than towards Y.* And the reverse will be the case if $m < n$. The common hyperbola approaches both asymptotes equally.

Fig. 1.

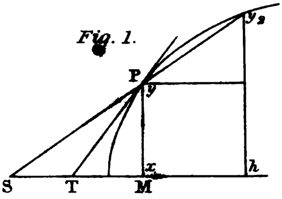


Fig. 3.

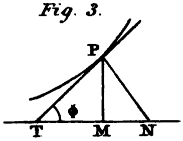
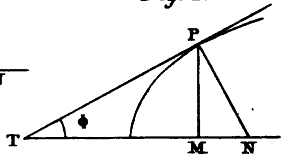


Fig. 2.



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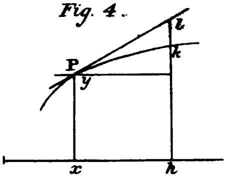
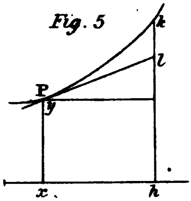


Fig. 5



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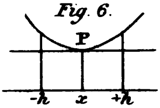

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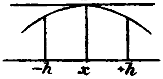

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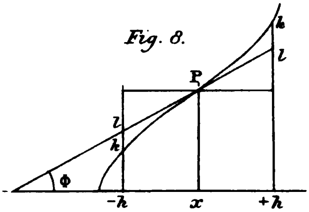

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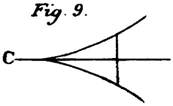


Fig. 10.

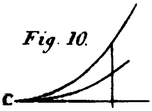
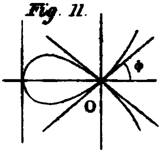


Fig. 11.



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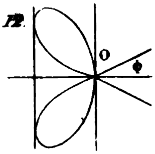

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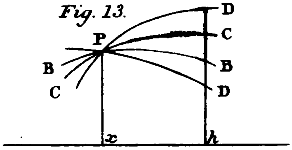

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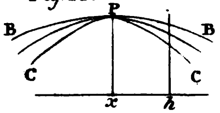


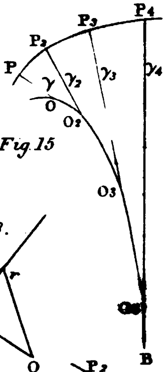
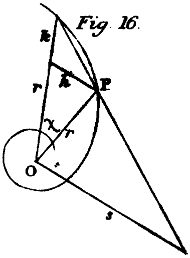


Fig 15

Fig 16.



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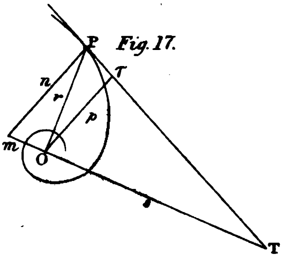

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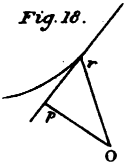
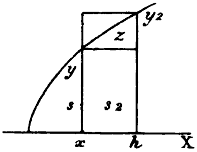


Fig 19



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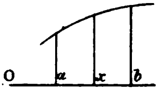

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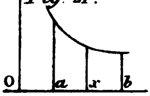

Fig. 22.